# MA 137 — Calculus 1 with Life Science Applications
## The Definite Integral
(Section 6.1)

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## Some Properties of Definite Integrals

1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right)$

4. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

5. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

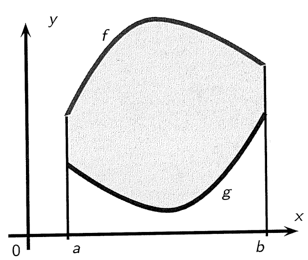
6. If $m \le f(x) \le M$ on $[a,b]$ then
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$





## Geometric Illustration of Some of the Properties

Property **3.** says that if $f$ and $g$ are two positive valued functions with $f$ greater than $g$, then

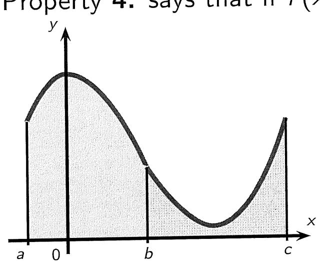



$$\int_a^b (f(x) - g(x))\,dx$$

gives the area between the graphs of $f$ and $g$

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Property **4.** says that if $f(x)$ is a positive valued function then the area underneath the graph of $f(x)$ between $a$ and $b$ plus the area underneath the graph of $f(x)$ between $b$ and $c$ equals the area underneath the graph of $f(x)$ between $a$ and $c$.



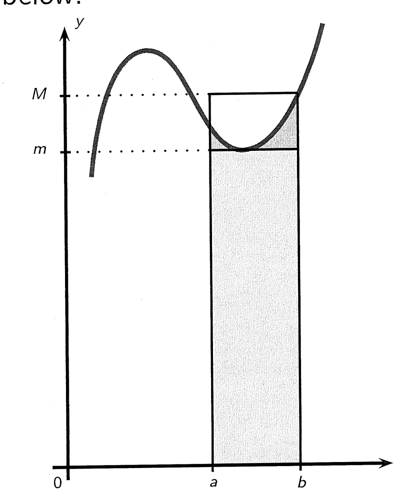





Property **5.** follows from Properties **4.** and **1.** by letting $c = a$.

$$0 = \int_a^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx.$$

Property **6.** is illustrated in the picture below.



## Example 1: (Online Homework, HW23, # 8)

The sum

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

can be written as a single definite integral of the form

$$\int_{a}^{b} f(x)\,dx$$

for appropriate $a$ and $b$. Determine these values.



$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \underbrace{\int_{-2}^{-1} f(x)\,dx}_{+\int_{-1}^{-2} f(x)\,dx}$$

because of property 5.

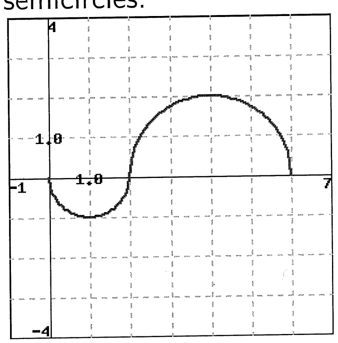
$$= + \int_{-1}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx$$

$$= \int_{-1}^{5} f(x)\,dx$$ because of property 4.



## Example 2: (Online Homework, HW23, # 5)

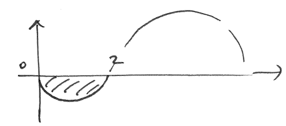
Evaluate the integrals for $f(x)$ shown in the figure below. The two parts of the graph are semicircles.




$$\int_0^2 f(x)\,dx \qquad \int_0^6 f(x)\,dx \qquad \int_1^4 f(x)\,dx \qquad \int_1^6 |f(x)|\,dx.$$

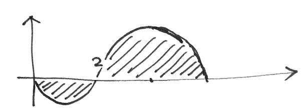


$\int_0^2 f(x)\,dx =$




$$= -\frac{\pi(1)^2}{2} = \boxed{-\pi/2} \cong -1.5708$$

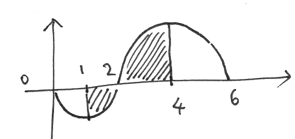
$\int_0^6 f(x)\,dx =$




$$= -\frac{\pi(1)^2}{2} + \frac{\pi(2)^2}{2} = -\pi/2 + 2\pi = \boxed{\frac{3}{2}\pi}$$

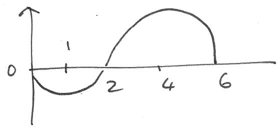
$$\cong 4.7124$$

$\int_1^4 f(x)\,dx =$

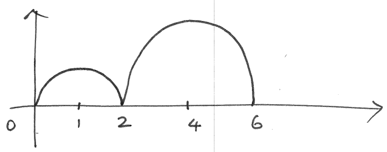



$$= -\frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{4} = -\frac{\pi}{4} + \pi = \boxed{\frac{3}{4}\pi} \cong 2.3562$$

If $f$ has the graph

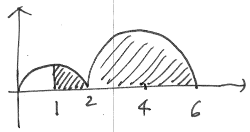



then $|f|$ has the following graph




Hence

$$\int_1^6 |f(x)|\,dx =$$




$$= + \frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{2} = \frac{\pi}{4} + 2\pi = \boxed{\frac{9}{4}\pi}$$

$$\cong 7.0686$$



## Example 3: (Neuhauser, Problem # 37, p. 320)

Use an area formula from geometry to find the value of the integral below

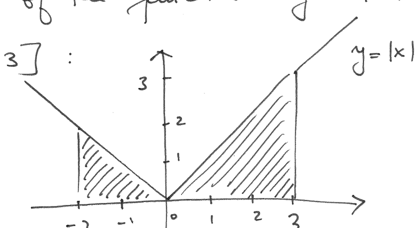
$$\int_{-2}^{3} |x|\,dx$$

by interpreting it as the (signed) area under the graph of an appropriately chosen function.



$$\int_{-2}^{3} |x|\,dx$$

Let's look at the graph of the function $y = |x|$ over the interval $[-2, 3]$:




Hence $\int_{-2}^{3} |x|\,dx$ gives the area of the 2 shaded regions (which are triangles):

$$\int_{-2}^{3} |x|\,dx = \frac{2\cdot 2}{2} + \frac{3\cdot 3}{2} = \frac{13}{2} = \underline{\underline{6.5}}$$



## Example 4: (Neuhauser, Problem # 41, p. 320)

Use an area formula from geometry to find the value of the integral below

$$\int_{-2}^{2} \left(\sqrt{4 - x^2} - 2\right) dx$$

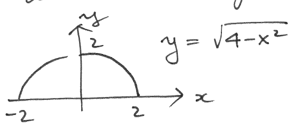
by interpreting it as the (signed) area under the graph of an appropriately chosen function.

$$\int_{-2}^{2}\left(\sqrt{4-x^2}-2\right)dx$$

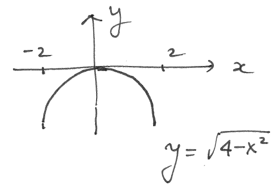
Let's graph the function

$$y=\sqrt{4-x^2}-2 \quad \text{on } [-2,2]$$

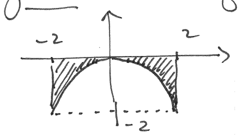
Notice that $y=\sqrt{4-x^2}$ by itself is the graph of the upper half of the circle of radius 2 centered at the origin:




hence




$$\int_{-2}^{2}\left(\sqrt{4-x^2}-2\right)dx = \text{"signed" area of the region}$$




$$\simeq \boxed{-1.7168}$$

$$= -\left[\text{area rectangle} - \text{area semicircle}\right] = -\left[4\cdot 2 - \frac{\pi\, 2^2}{2}\right]$$



## Example 5: (Neuhauser, Problem # 50(c),(f), p. 320)

Given that

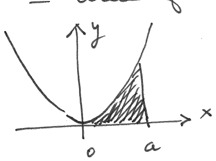
$$\int_0^a x^2\,dx = \frac{1}{3}a^3$$

evaluate the following

$$\int_{-1}^{3}\frac{1}{3}x^2\,dx \qquad \int_2^4 (x-2)^2\,dx.$$

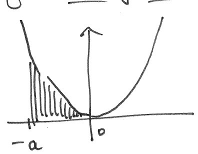


$$\int_0^a x^2\,dx = \frac{1}{3}a^3 = \text{area of the shaded region}$$




By symmetry of the function we also have that

$$\int_{-a}^{0} x^2\,dx =$$




$$= \frac{1}{3}a^3$$

positive sign

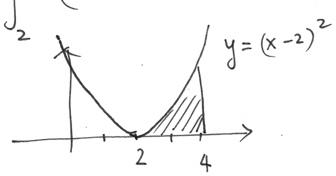
Hence:

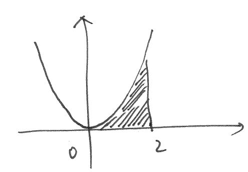
$$\int_{-1}^{3}\frac{1}{3}x^2\,dx = \frac{1}{3}\int_{-1}^{3}x^2\,dx = \frac{1}{3}\left[\int_{-1}^{0}x^2\,dx + \int_0^3 x^2\,dx\right]$$

$$= \frac{1}{3}\left[\frac{1}{3}\cdot 1^2 + \frac{1}{3}\cdot 3^3\right] = \frac{1}{3}\left[\frac{1}{3}+3\right] = \boxed{\frac{10}{9}}$$

the graph of $y=(x-2)^2$ is obtained from the graph of $y=x^2$ by shifting it of 2 units to the right; hence:

$$\int_2^4 (x-2)^2\,dx \quad \text{must be the same as} \quad \int_0^2 x^2\,dx$$




Hence $$\int_2^4 (x-2)^2\,dx = \int_0^2 x^2\,dx = \frac{1}{3}2^3 = \boxed{\frac{8}{3}}$$