MA 137 — Calculus 1 with Life Science Applications
The Fundamental Theorem of Calculus
(Section 6.2)

Department of Mathematics
University of Kentucky

An Example

- The easiest procedure for computing definite integrals is not by computing a limit of a Riemann sum, but by relating integrals to (anti)derivatives.
- This relationship is so important in Calculus that the theorem that describes it is called

  the Fundamental Theorem of Calculus.
- We introduce the theorem by first analyzing a simple example.

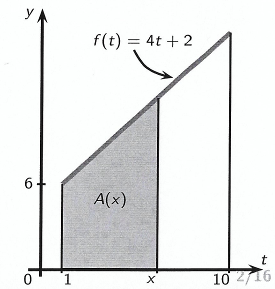
Example 1

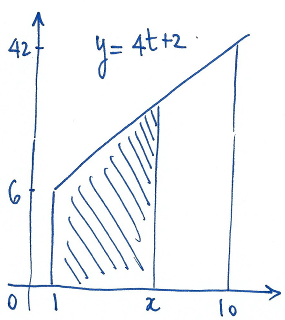
Find a formula for $A(x) = \int_1^x (4t+2)\,dt,$

where $1 \leq x \leq 10$.

Find the values $A(5)$, $A(10)$, and $A(1)$.

What is the derivative of $A(x)$ with respect to x?





$A(x) = \int_1^x (4t+2)\,dt$ = area of the shaded trapezoid

Bases: 6 and $4x+2$

height: $x-1$

Hence $A(x) = \frac{[6 + (4x+2)]\cdot(x-1)}{2}$

$\therefore\ A(x) = \int_1^x (4t+2)\,dt = \frac{(4x+8)(x-1)}{2} = (2x+4)(x-1)$

$= 2x^2 + 2x - 4$

$A(1) = 2\cdot 1^2 + 2\cdot 1 - 4 = \underline{\underline{0}}$

$A(5) = 2\cdot 5^2 + 2\cdot 5 - 4 = \underline{\underline{56}}$

$A(10) = 2\cdot 10^2 + 2\cdot 10 - 4 = \underline{\underline{216}}$

$\boxed{\frac{d}{dx} A(x) = \frac{d}{dx}(2x^2+2x-4) = 4x+2}$

The Main Idea of the FTC

Suppose that for *any* function $f(t)$ it were true that the area function

$$A(x) = \int_a^x f(t)\,dt$$

satisfies

$$A(a) = \int_a^a f(t)\,dt = 0 \qquad A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Moreover, suppose that $B(x)$ is any function such that: $B'(x) = f(x) = A'(x)$. By a consequence of the MVT we know that there is a constant value c such that $B(x) = A(x) + c$.

All these facts put together help us easily evaluate $\int_a^b f(t)\,dt$.

Indeed
$$\begin{aligned} \int_a^b f(t)\,dt &= A(b) = A(b) - 0 \\ &= A(b) - A(a) = [A(b)+c] - [A(a)+c] \\ &= B(b) - B(a) \end{aligned}$$

The FTC

The previous 'speculations' are actually true for any continuous function on the interval over which we are integrating. These results are stated in the following theorem, which is divided into two parts:

Theorem (The Fundamental Theorem of Calculus)

PART I: *Let $f(t)$ be a continuous function on the interval $[a,b]$. Then the function $A(x)$, defined by the formula*

$$A(x) = \int_a^x f(t)\,dt$$

for all x in the interval $[a,b]$, is an antiderivative of $f(x)$, that is

$$A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x in the interval $[a,b]$.

PART II: *Let $F(x)$ be any antiderivative of $f(x)$ on $[a,b]$, so that $F'(x) = f(x)$ for all x in the interval $[a,b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Special Notation for Part II:

Part II of the FTC tells us that evaluating a definite integral is a two-step process:

- find *any* antiderivative $F(x)$ of the function $f(x)$; and then
- compute the difference $F(b) - F(a)$.

A notation has been devised to separate the two steps of this process: $F(x)\Big|_a^b$ stands for the difference $F(b) - F(a)$. Thus

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

About the Proof of the FTC:

We already gave an explanation of why the second part of the Fundamental Theorem of Calculus follows from the first one.
To prove the first part we need to use the definition of the derivative.

Proof of Part I

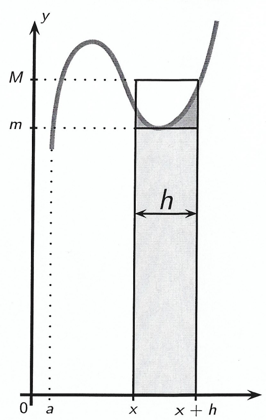
We must show that

$$A'(x) = \lim_{h\to 0}\frac{A(x+h) - A(x)}{h} = f(x).$$

For convenience, let us assume that f is a positive valued function. Given that $A(x)$ is defined by $\int_a^x f(t)\,dt$, the numerator of the above difference quotient is

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

Using properties **4.** and **5.** of definite integrals, the above difference equals $\int_x^{x+h} f(t)\,dt$.



As the function f is continuous over the interval $[x, x+h]$, the Extreme Value Theorem says that there are values c_1 and c_2 in $[x, x+h]$ where f attains the minimum and maximum values, say m and M, respectively.

Thus $m \le f(t) \le M$ on $[x, x+h]$. As the length of the interval $[x, x+h]$ is h, by property **6.** of definite integrals we have that

$$f(c_1)h = mh \le \int_x^{x+h} f(t)\,dt \le Mh = f(c_2)h$$

or, equivalently,

$$f(c_1) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le f(c_2).$$

Finally, as f is continuous we have that $\lim_{h\to 0} f(c_1) = f(x) = \lim_{h\to 0} f(c_2)$.

This concludes the proof.

Example 2: (Online Homework HW24, # 2)

Suppose

$$f(x) = \int_0^x \frac{t^2 - 16}{2 + \cos^2(t)}\, dt$$

For what value(s) of x does $f(x)$ have a local maximum?

$$f(x) = \int_0^x \frac{t^2-16}{2+\cos^2(t)}\, dt$$

By the FTC part I :

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \int_0^x \frac{t^2-16}{2+\cos^2(t)}\, dt = \frac{x^2-16}{2+\cos^2(x)}$$

Hence $f'(x) = 0 \iff x^2 - 16 = 0 \iff x = \pm 4$

Now you can use test points to determine the sign of $f'(x)$:

$$\underset{-4}{++++} \quad ------ \quad \underset{4}{} \quad ++++$$

f inc. f dec. f inc.

$\therefore$ f has a local max at $\boxed{x=-4}$

Example 3: (Online Homework HW24, # 6)

Find a function f and a number a such that

$$2 + \int_a^x \frac{f(t)}{t^7}\, dt = 4x^{-3}$$

$$2 + \int_a^x \frac{f(t)}{t^7}\, dt = 4x^{-3}$$

Notice that if we plug in $\boxed{x=a}$ in the above equation we get :

$$2 + \underbrace{\int_a^a \frac{f(t)}{t^7}\, dt}_{=0} = 4a^{-3} \qquad \therefore \quad 2 = \frac{4}{a^3}$$

as the interval of integration has length 0

$$\text{or} \quad a^3 = \frac{4}{2}$$

$$\therefore \quad \boxed{a = \sqrt[3]{2}}$$

Hence we obtain so far :

$$\boxed{2 + \int_{\sqrt[3]{2}}^x \frac{f(t)}{t^7}\, dt = 4x^{-3}}$$

In order to find f, let's take the derivative of both sides with respect to x, and let's apply the FTC Part I:

$$\frac{d}{dx}\left[2 + \int_{\sqrt[3]{2}}^{x} \frac{f(t)}{t^7}\,dt\right] = \frac{d}{dx}\, 4x^{-3}$$

$$\Longrightarrow \quad \frac{f(x)}{x^7} = 4(-3)\cdot x^{-4}$$

Hence $\quad f(x) = -12\, x^{-4}\cdot x^7$

$$\therefore \quad \boxed{f(x) = -12\,x^3}$$

Leibniz's Rule

Combining the chain rule and the FTC (Part I), we can differentiate integrals with respect to x when the upper and/or lower limits of integration are function of x.

We summarize these facts into the following result:

Leibniz's Rule

If $g(x)$ and $h(x)$ are differentiable functions and $f(u)$ is continuous for u between $g(x)$ and $h(x)$, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f[h(x)]h'(x) - f[g(x)]g'(x)$$

Example 4:

Compute

$$\frac{d}{dx}\int_{\sin x}^{1} u^2\,du$$

$$\frac{d}{dx}\int_{\sin x}^{1} u^2\,du = \frac{d}{dx}\left[-\int_{1}^{\boxed{\sin x}} u^2\,du\right] =$$

Set $\boxed{w = \sin x}$ then by the chain rule we have

$$= -\left[\frac{d}{dw}\int_{1}^{w} u^2\,du\right]\cdot\left[\frac{dw}{dx}\right] = -w^2\cdot\frac{dw}{dx}$$

$$= -(\sin x)^2\cdot\cos x = \boxed{-\sin^2 x\cdot\cos x}$$

This confirms the Leibniz's Rule we described earlier.

Example 5: (Online Homework HW24, # 5)

Find the derivative of the following function

$$F(x) = \int_{x^4}^{x^6} (2t-1)^3\, dt$$

using the Fundamental Theorem of Calculus.

$$F(x) = \int_{x^4}^{x^6} (2t-1)^3\, dt = \int_{x^4}^{0} (2t-1)^3 dt + \int_{0}^{x^6} (2t-1)^3 dt$$

for example

$$= \int_0^{x^6} (2t-1)^3 dt - \int_0^{x^4} (2t-1)^3 dt$$

now apply FTC Part I together with the chain rule

$$F'(x) = \frac{d}{dx}\int_0^{x^6} (2t-1)^3 dt - \frac{d}{dx}\int_0^{x^4} (2t-1)^3 dt$$

$$= (2x^6-1)^3 \cdot \underline{6x^5} - (2x^4-1)^3 \cdot \underline{4x^3}$$

which confirms the Leibniz's Rule we described earlier

Example 6: (Online Homework HW24, # 7)

Evaluate the definite integral

$$\int_4^7 \left(\frac{d}{dt}\sqrt{3+3t^4}\right) dt$$

using the Fundamental Theorem of Calculus.

Notice that

$$\int \left(\frac{d}{dt}\sqrt{3+3t^4}\right) dt = \sqrt{3+3t^4} + C$$

as the process of antidifferentiation is the inverse of the process of differentiation.

Hence:

$$\int_4^7 \left(\frac{d}{dt}\sqrt{3+3t^4}\right) dt = \sqrt{3+3t^4}\Big]_4^7$$

we choose the constant to be $\underline{C=0}$

$$= \sqrt{3+3\cdot 7^4} - \sqrt{3+3\cdot 4^4} = \sqrt{7206} - \sqrt{771}$$

$$= 84.8882 - 27.7669 \cong 57.1213$$

Example 7: (Online Homework HW24, # 12)

Evaluate the definite integral

$$\int_1^4 \frac{x^2+5}{x}\,dx$$

$\int_1^4 \frac{x^2+5}{x}\,dx$ We use FTC Part 2

We first need an antiderivative of $\frac{x^2+5}{x}$:

$$\int \frac{x^2+5}{x}\,dx = \int \left(x + \frac{5}{x}\right) dx = \int x\,dx + 5\int \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2 + 5\ln|x| + C$$

Now we have :

$$\int_1^4 \frac{x^2+5}{x}\,dx = \frac{1}{2}x^2 + 5\ln|x| + C\Big]_1^4 =$$

$$= \left[\frac{1}{2}4^2 + 5\ln(4) + \cancel{C}\right] - \left[\frac{1}{2}1^2 + 5\ln(1) + \cancel{C}\right]$$

$$= 8 + 5\ln 4 - \tfrac{1}{2} = \boxed{\tfrac{15}{2} + 5\ln 4} \cong 14.4315$$

Example 8: (Online Homework HW24, # 14)

Evaluate the definite integral

$$\int_0^1 (x^2 + 8 - 2e^{-2x})\,dx$$

$$\int (x^2 + 8 - 2e^{-2x})\,dx =$$

$$\int x^2\,dx + 8\int 1\cdot dx + \int -2e^{-2x}\,dx$$

$$= \frac{1}{3}x^3 + 8x + e^{-2x} + C$$

Hence :

$$\int_0^1 (x^2 + 8 - 2e^{-2x})\,dx = \frac{1}{3}x^3 + 8x + e^{-2x}\Big]_0^1$$

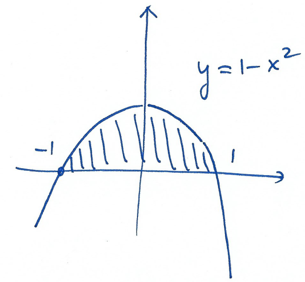
we need just one antiderivative
we choose the one with $C=0$

$$= \left[\frac{1}{3}\cdot 1^3 + 8\cdot 1 + e^{-2\cdot 1}\right] - \left[\frac{1}{3}\cdot 0^3 + 8\cdot 0 + e^{-2\cdot 0}\right]$$

$$= \frac{1}{3} + 8 + e^{-2} - 1 = \frac{22}{3} + e^{-2} \cong 7.4687$$

Example 9: (Online Homework HW24, # 15)

Find the area bounded by the function $y = 1 - x^2$ and the x-axis.



Notice that the intersections with the x-axis are given by

$$0 = 1 - x^2 \iff x = 1.$$

Hence we need to compute $\int_{-1}^{1} (1-x^2)\,dx$

Observe that by the symmetry of the function ($\because f(x)$ is even) then

$$\int_{-1}^{1} (1-x^2)\,dx = 2\int_{0}^{1} (1-x^2)\,dx = 2\left(x - \tfrac{1}{3}x^3\right)\Big]_0^1$$

$$= \left[2\left(1 - \tfrac{1}{3}\right)\right] - [0] = \boxed{4/3} \text{ units}$$