2. (**Gompertz Growth Model)** The Gompertz growth curve is sometimes used to study the growth of populations. Its properties are quite similar to the properties of the logistic growth curve. The Gompertz growth curve is given by

$$N(t) = K\exp\left(-ae^{-bt}\right)$$

for $t \geq 0$, where K and b are positive constants.

(a) Show that $N(0) = Ke^{-a}$ and, hence,

$$a = \ln\frac{K}{N_0}$$

if $N_0 = N(0)$.

(b) Show that $y = K$ is a horizontal asymptote and that $N(t) < K$ if $N_0 < K$, $N(t) = K$ if
$N_0 = K$, and $N(t) > K$ if $N_0 > K$.

(c) Show that

$$\frac{dN}{dt} = bN(\ln K - \ln N)\text{and } \frac{d^2N}{dt^2} = b\frac{dN}{dt}\left(\ln K - \ln N - 1\right)$$

(d) Use your results in (b) and (c) to show that $N(t)$ is strictly increasing if $N_0 < K$ and strictly decreasing if $N_0 > K$.

(e) When does $N(t)$, $t \geq 0$, have an inflection point? Discuss its concavity.

(f) Graph $N(t)$ when $K = 100$ and $b = 1$ if

(i) $N_0 = 20$,
(ii) $N_0 = 70$, and
(iii) $N_0 = 150$,
and compare your graphs with your answers in (b)-(e).