FastTrack — MA 137/MA 113 — BioCalculus

Functions (3): The Algebra of Functions

Alberto Corso – ⟨alberto.corso@uky.edu⟩

Department of Mathematics – University of Kentucky

Goal: We learn how two functions can be combined to form new functions. We then define one-to-one functions, which allows us to introduce the notion of inverse of a one-to-one function. These topics are of importance when we study exponential and logarithmic functions.

Combining functions

Let f and g be functions with domains A and B. We define new functions $f+g$, $f-g$, fg, and f/g as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{Domain } A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Domain } \{x \in A \cap B \,|\, g(x) \neq 0\}$$

Note

Consider the above definition $(f+g)(x) = f(x)+g(x)$.

The $+$ on the left hand side stands for the operation of addition of functions.

The $+$ on the right hand side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$.

Similar remarks hold true for the other definitions.

Example 1:

Let us consider the functions $f(x) = x^2 - 2x$ and $g(x) = 3x - 1$.

Find $f + g$, $f - g$, fg, and f/g and their domains.

Example 2:

Let us consider the functions $f(x) = \sqrt{9 - x^2}$ and $g(x) = \sqrt{x^2 - 1}$.

Find $f + g$, $f - g$, fg, and f/g and their domains.

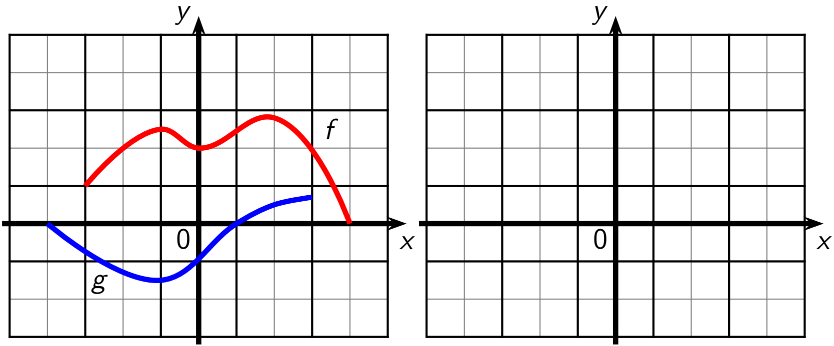
The graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**.

This means that to obtain the value of $f + g$ at any point x we add the corresponding values of $f(x)$ and $g(x)$, that is, the corresponding y-coordinates.

Similar statements can be made for the other operations on functions.

Example 3:

Use graphical addition to sketch the graph of $f + g$.



graph of $f + g$

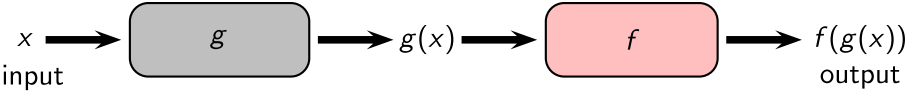
Composition of Functions

Given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.

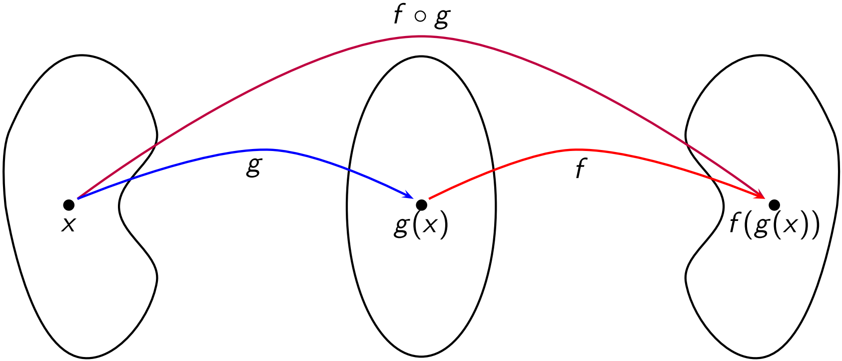
The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: 'f composed with g' or 'f after g')

$$\boxed{(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).}$$

WARNING: $f \circ g \neq g \circ f$.



Machine diagram of $f \circ g$



Arrow diagram of $f \circ g$

Example 4:

Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate:

$f(g(0)) =$

$g(f(0)) =$

$f(f(4)) =$

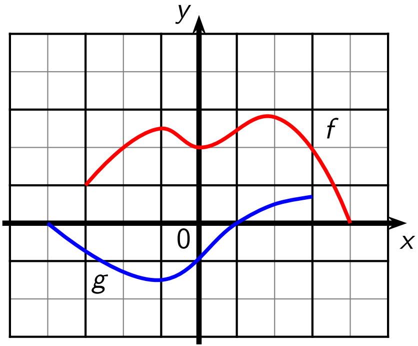
$(g \circ g)(2) =$

$(f \circ g)(x) =$

$(g \circ f)(x) =$

Example 5:

Let f and g be the functions considered in Example 3. Use the information provided by the graphs of f and g to find $f(g(1))$, $g(f(0))$, $f(g(0))$, and $g(f(4))$.



Example 6:

Let $f(x) = \dfrac{x}{x+1}$ and $g(x) = 2x - 1$.

Find the functions $f \circ g$, $g \circ f$, and $f \circ f$ and their domains.

Example 7:

Express the function $F(x) = \dfrac{x^2}{x^2 + 4}$ in the form $F(x) = f(g(x))$.

Example 8:

Find functions f and g so that $f \circ g = H$ if $H(x) = \sqrt[3]{2 + \sqrt{x}}$.

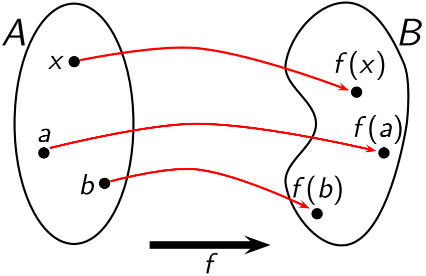
Definition of a One-One Function

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$



Horizontal Line Test

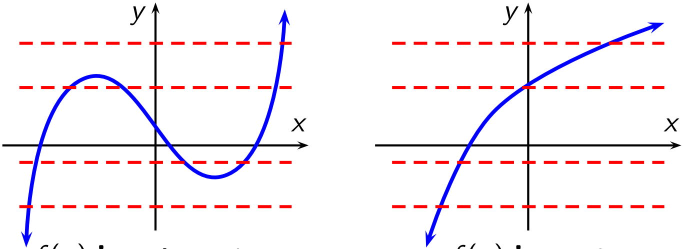
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test

A function is one-to-one

$\Longleftrightarrow$

no horizontal line intersects its graph more than once.



$f(x)$ **is not** one-to-one $\qquad$ $f(x)$ **is** one-to-one

Example 9:

Show that the function $f(x) = 5 - 2x$ is one-to-one.

Example 10:

Graph the function $f(x) = (x-2)^2 - 3$. The function is not one-to-one: Why? Can you restrict its domain so that the resulting function is one-to-one? (There is more than one correct answer.)

The Inverse of a Function

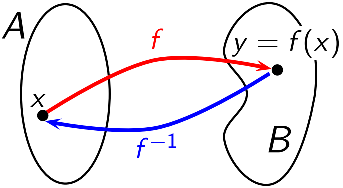
One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Its **inverse function** f^{-1} has domain B and range A and is defined by

$$\boxed{f^{-1}(y) = x \quad \iff \quad f(x) = y,}$$

for any $y \in B$.



If f takes x to y,
then f^{-1} takes y back to x.
I.e., f^{-1} undoes what f does.

NOTE:
f^{-1} does NOT mean $\frac{1}{f}$.

Example 11:

Suppose $f(x)$ is a one-to-one function.

If $f(2) = 7$, $f(3) = -1$, $f(5) = 18$, $f^{-1}(2) = 6$ find:

$f^{-1}(7) =$ $\qquad$ $f(6) =$

$f^{-1}(-1) =$ $\qquad$ $f(f^{-1}(18)) =$

If $g(x) = 9 - 3x$, then $g^{-1}(3) =$

Properties of Inverse Functions

Let $f(x)$ be a one-to-one function with domain A and range B. The inverse function $f^{-1}(x)$ satisfies the following "cancellation" properties:

$$\boxed{f^{-1}(f(x)) = x \text{ for every } x \in A}$$

$$\boxed{f(f^{-1}(x)) = x \text{ for every } x \in B}$$

Conversely, any function $f^{-1}(x)$ satisfying the above conditions is the inverse of $f(x)$.

Example 12:

Show that the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverses of each other.

Example 13:

Show that the functions $f(x) = \dfrac{1+3x}{5-2x}$ and $g(x) = \dfrac{5x-1}{2x+3}$ are inverses of each other.

How to find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 14:

Find the inverse of $y = 4x - 7$.

Example 15:

Find the inverse of $y = \dfrac{1}{x+2}$.

Example 16:

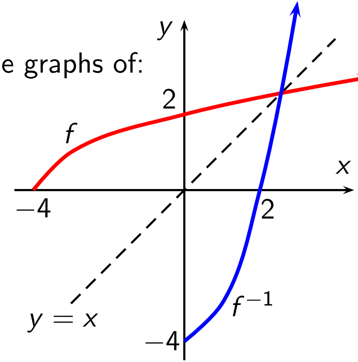
Find the inverse of $y = \dfrac{2-x}{x+2}$.

Graph of the Inverse Function

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**

The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4}$$
and
$$f^{-1}(x) = x^2 - 4, \ x \geq 0.$$



Example 17:

Find the inverse of the function $f(x) = 1 + \sqrt{1+x}$.
Find the domain and range of f and f^{-1}. Graph f and f^{-1} on the same cartesian plane.