FastTrack — MA 137/MA 113 — BioCalculus

Functions (3): The Algebra of Functions

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Goal: We learn how two functions can be combined to form new functions. We then define one-to-one functions, which allows us to introduce the notion of inverse of a one-to-one function. These topics are of importance when we study exponential and logarithmic functions.

Combining functions

Let f and g be functions with domains A and B. We define new functions $f+g$, $f-g$, fg, and f/g as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{Domain } A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

Note

Consider the above definition $(f+g)(x) = f(x)+g(x)$.

The $+$ on the left hand side stands for the operation of addition of functions.

The $+$ on the right hand side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$.

Similar remarks hold true for the other definitions.

Example 1:

Let us consider the functions $f(x) = x^2 - 2x$ and $g(x) = 3x - 1$.

Find $f+g$, $f-g$, fg, and f/g and their domains.

$$(f+g)(x) = f(x) + g(x) = (x^2 - 2x) + (3x - 1)$$
$$= x^2 + x - 1$$

$$(f-g)(x) = f(x) - g(x) = (x^2 - 2x) - (3x - 1)$$
$$= x^2 - 5x + 1$$

$$(fg)(x) = f(x) \cdot g(x) = (x^2 - 2x)(3x - 1)$$
$$= 3x^3 - 7x^2 + 2x$$

domain: all real numbers

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 2x}{3x - 1}$$

domain : $\{ x \in \mathbb{R} \mid x \neq 1/3 \}$

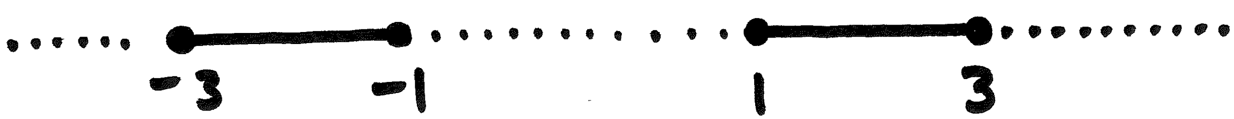
Example 2:

Let us consider the functions $f(x) = \sqrt{9 - x^2}$ and $g(x) = \sqrt{x^2 - 1}$.

Find $f + g$, $f - g$, fg, and f/g and their domains.

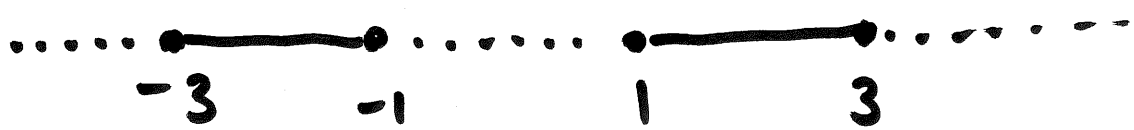
$$(f+g)(x) = f(x) + g(x) = \sqrt{9-x^2} + \sqrt{x^2-1}$$

domain:

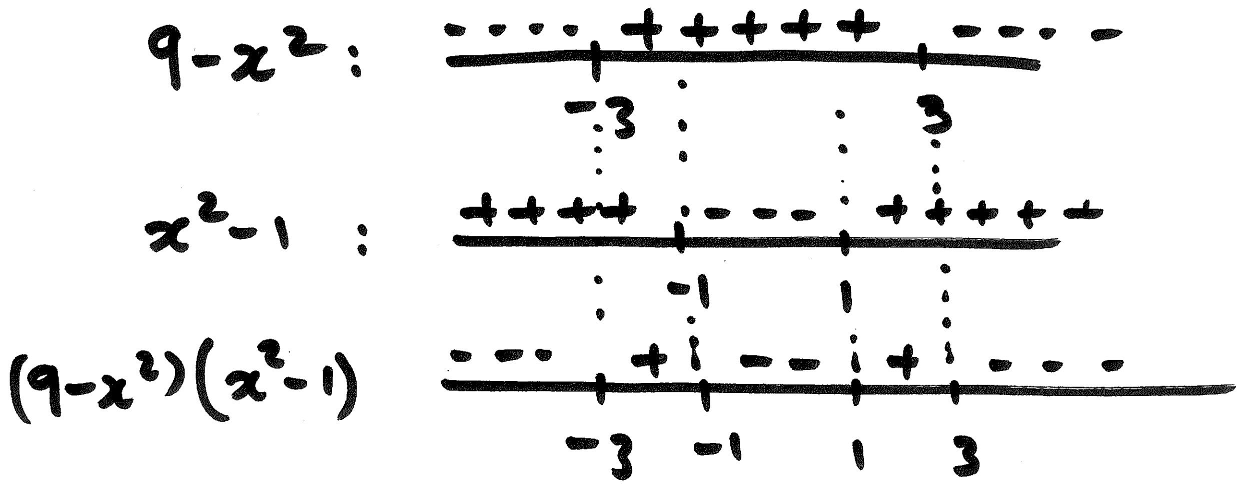


$$(f-g)(x) = f(x) - g(x) = \sqrt{9-x^2} - \sqrt{x^2-1}$$

domain:



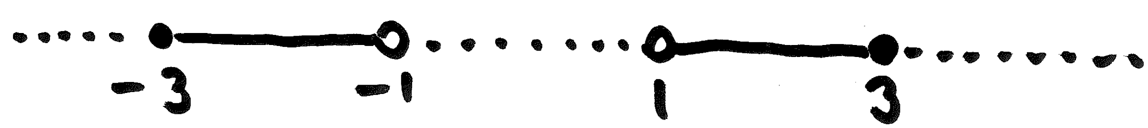
$$(fg)(x) = \sqrt{9-x^2} \cdot \sqrt{x^2-1} = \sqrt{(9-x^2)(x^2-1)}$$



$\therefore$ domain : $\quad -3 \le x \le -1$ and $1 \le x \le 3$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}} = \sqrt{\frac{9-x^2}{x^2-1}}$$

domain :



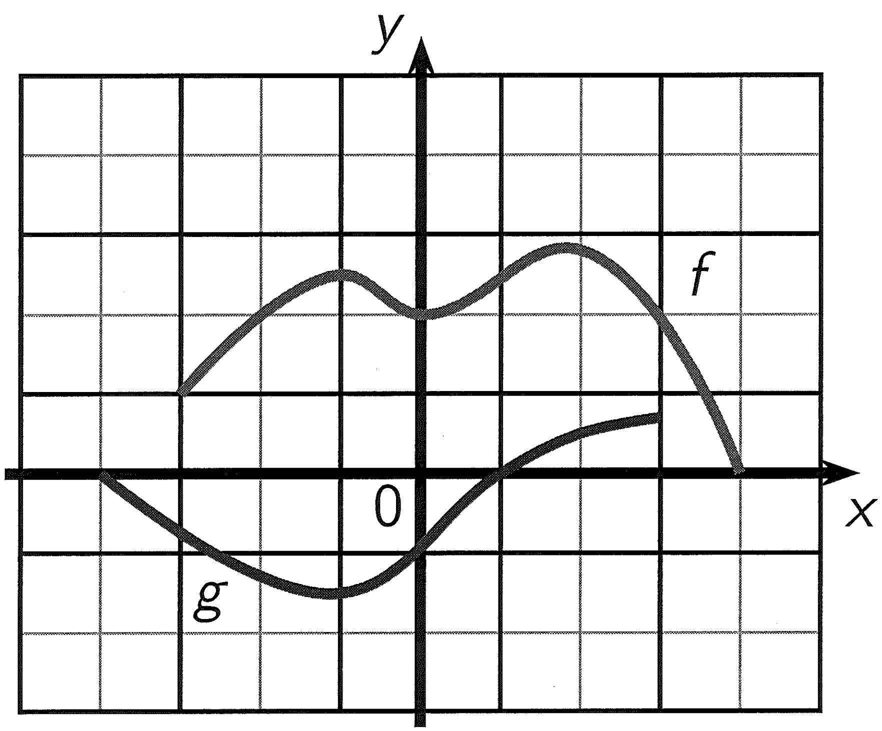
The graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**.

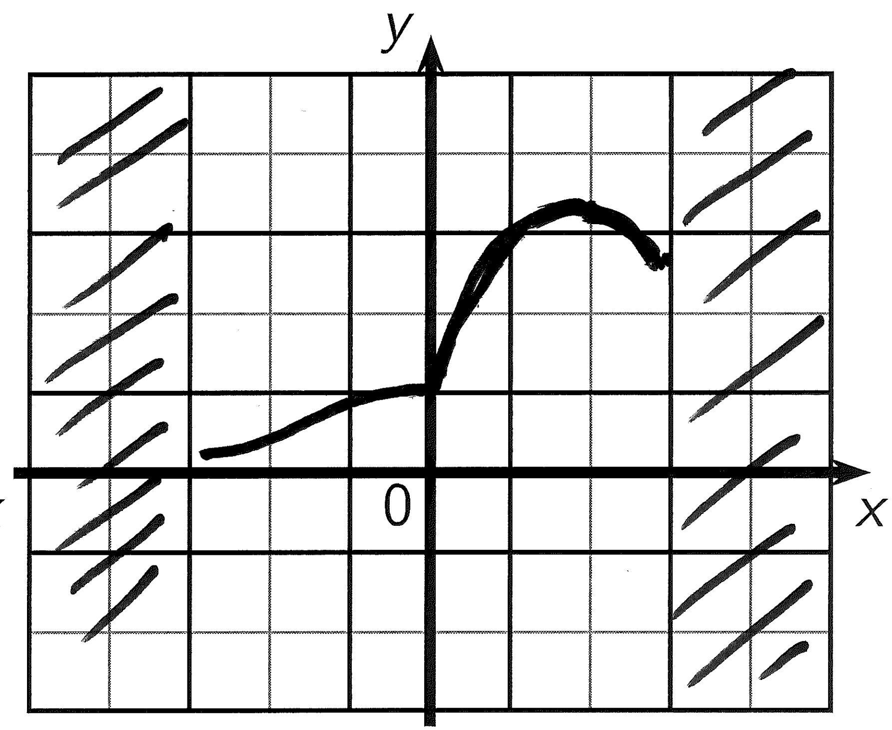
This means that to obtain the value of $f + g$ at any point x we add the corresponding values of $f(x)$ and $g(x)$, that is, the corresponding y-coordinates.

Similar statements can be made for the other operations on functions.

Example 3:

Use graphical addition to sketch the graph of $f + g$.





graph of $f + g$

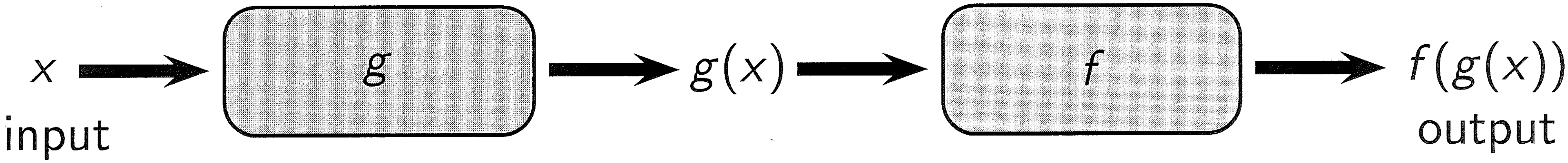
Composition of Functions

Given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.

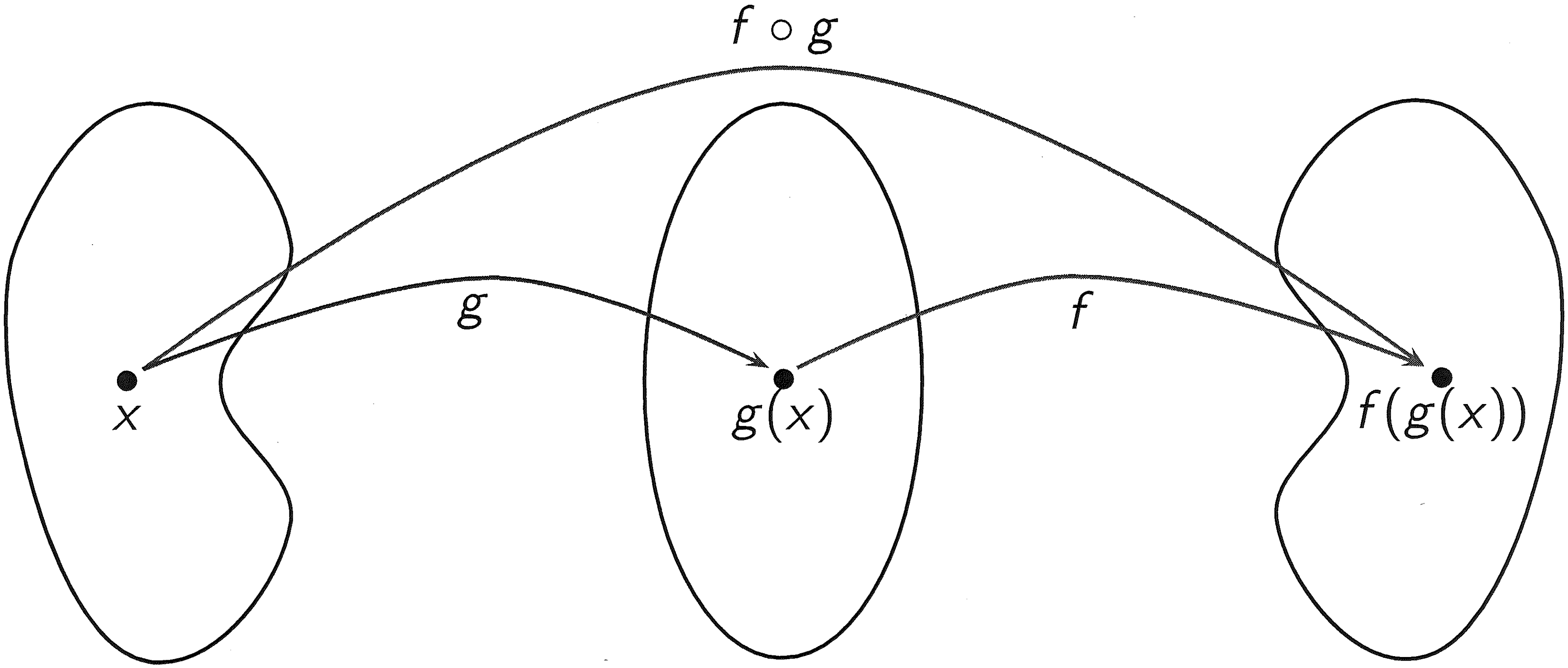
The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: 'f composed with g' or 'f after g')

$$\boxed{(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).}$$

WARNING: $f \circ g \neq g \circ f$.



Machine diagram of $f \circ g$



Arrow diagram of $f \circ g$

Example 4:

Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate:

$f(g(0)) = 3\,g(0) - 5 = 3 \cdot 2 - 5 = 1$

$g(f(0)) = 2 - [f(0)]^2 = 2 - [-5]^2 = -23$

$f(f(4)) = 3(f(4)) - 5 = 3(3 \cdot 4 - 5) - 5 = 16$

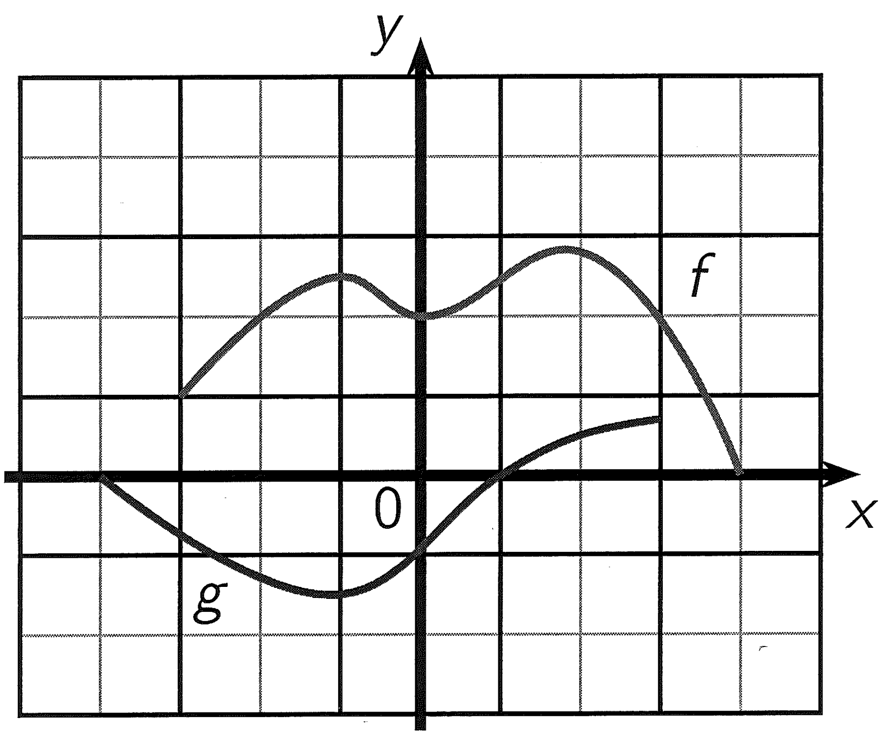
$(g \circ g)(2) = 2 - [g(2)]^2 = 2 - [-2]^2 = -2$

$(f \circ g)(x) = f(g(x)) = 3g(x) - 5 = 3(2 - x^2) - 5 = \boxed{1 - 3x^2}$

$(g \circ f)(x) = 2 - [f(x)]^2 = 2 - [3x - 5]^2 = \boxed{-9x^2 + 30x - 23}$

Example 5:

Let f and g be the functions considered in Example 3. Use the information provided by the graphs of f and g to find $f(g(1))$, $g(f(0))$, $f(g(0))$, and $g(f(4))$.



$g(1)=0 \Rightarrow f(g(1)) = \boxed{2}$

$f(0)=2 \Rightarrow g(f(0)) = \boxed{0.5}$

$g(0)=-1 \Rightarrow f(g(0)) = \boxed{2.5}$

$g(f(4)) = g(0) = \boxed{-1}$

Example 6:

Let $f(x) = \dfrac{x}{x+1}$ and $g(x) = 2x - 1$.

Find the functions $f \circ g$, $g \circ f$, and $f \circ f$ and their domains.

$$(f \circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)+1} = \frac{2x-1}{(2x-1)+1} = \boxed{\frac{2x-1}{2x}}$$

<u>domain</u> : $x \neq 0$

$$(g \circ f)(x) = g(f(x)) = 2\,f(x) - 1 = 2 \cdot \frac{x}{x+1} - 1$$

$$= \frac{2x}{x+1} - 1 = \frac{2x - (x+1)}{x+1} = \boxed{\frac{x-1}{x+1}}$$

<u>domain</u>: $x \neq -1$

$$(f \circ f)(x) = f(f(x)) = \frac{f(x)}{f(x)+1} =$$

$$= \frac{x/x+1}{\frac{x}{x+1}+1} = \frac{x/x+1}{\frac{x+(x+1)}{x+1}} =$$

$$= \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{\cancel{x+1}} \cdot \frac{\cancel{x+1}}{2x+1}$$

$$= \boxed{\frac{x}{2x+1}}$$

<u>domain</u>: $x \neq -1/2$

Example 7:

Express the function $F(x) = \dfrac{x^2}{x^2+4}$ in the form $F(x) = f(g(x))$.

$$x \xmapsto{\;g\;} x^2 \xmapsto{\;f\;} \frac{x^2}{x^2+4}$$

<u>thus</u>: $g(x) = x^2$

$$f(x) = \frac{x}{x+4}$$

Example 8:

Find functions f and g so that $f \circ g = H$ if $H(x) = \sqrt[3]{2+\sqrt{x}}$.

$$x \xmapsto{g} 2+\sqrt{x} \xmapsto{f} \sqrt[3]{2+\sqrt{x}}$$

<u>thus</u>:

$$g(x) = 2+\sqrt{x}$$

$$f(x) = \sqrt[3]{x}$$

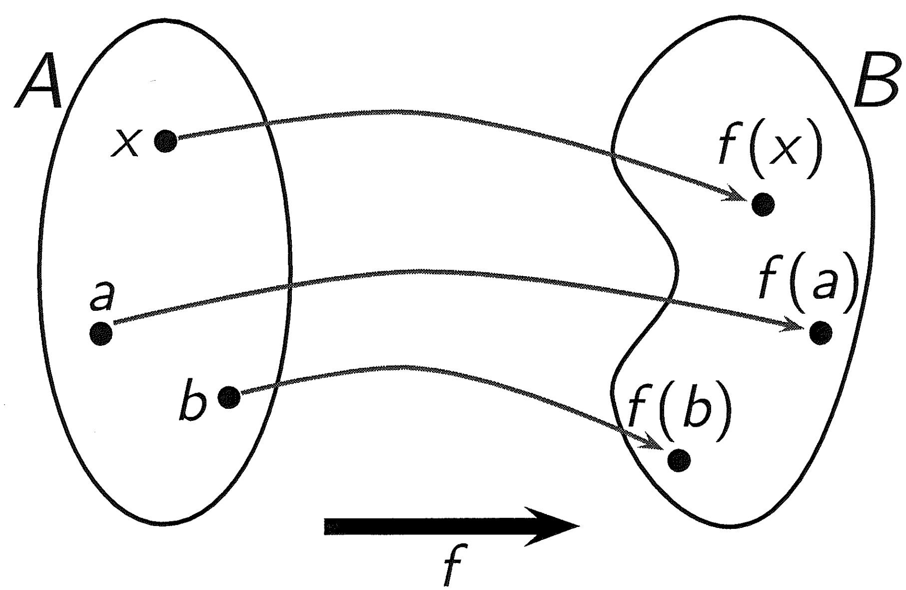
Definition of a One-One Function

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

$$\text{If } \; f(x_1) = f(x_2), \; \text{ then } \; x_1 = x_2.$$

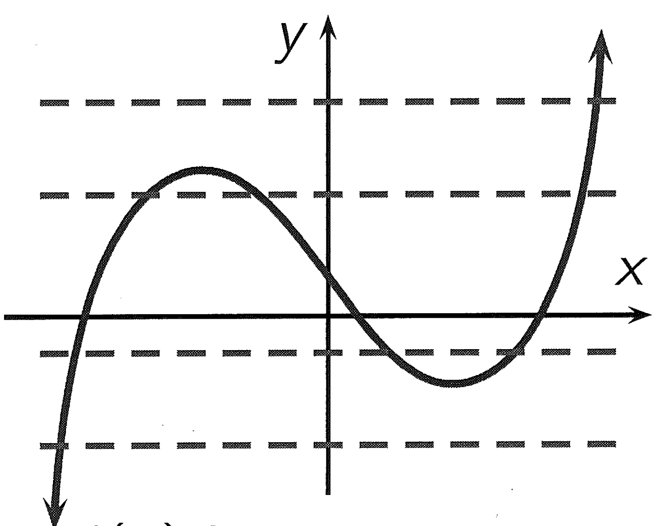


Horizontal Line Test

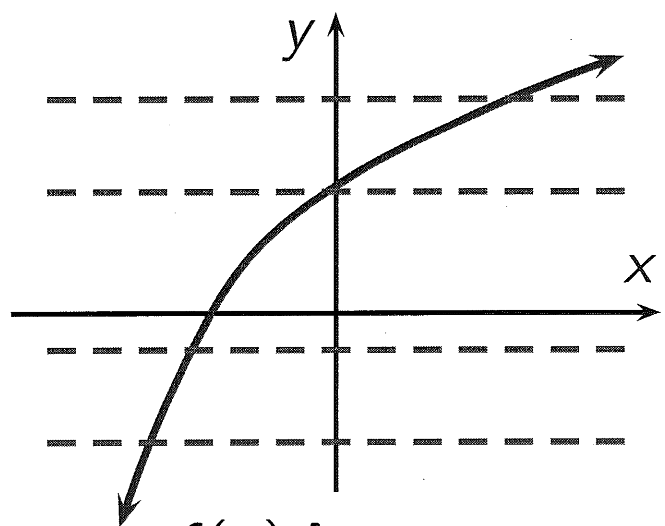
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test

A function is one-to-one
$$\Longleftrightarrow$$
no horizontal line intersects its graph more than once.



$f(x)$ **<u>is not</u>** one-to-one

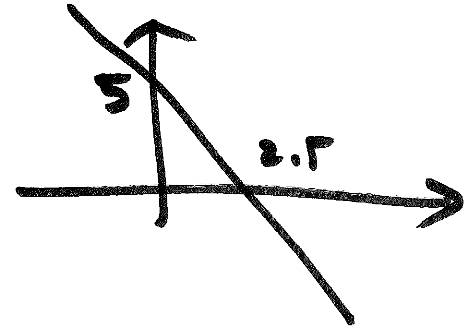


$f(x)$ **<u>is</u>** one-to-one

Example 9:

Show that the function $f(x) = 5 - 2x$ is one-to-one.

$\boxed{\text{1st method:}}$ Use the horizontal line test. The graph of f is a straight line and any hor. line intersect the graph in one point.



$\boxed{\text{2nd method:}}$ use the definition. Let $f(x_1) = f(x_2)$

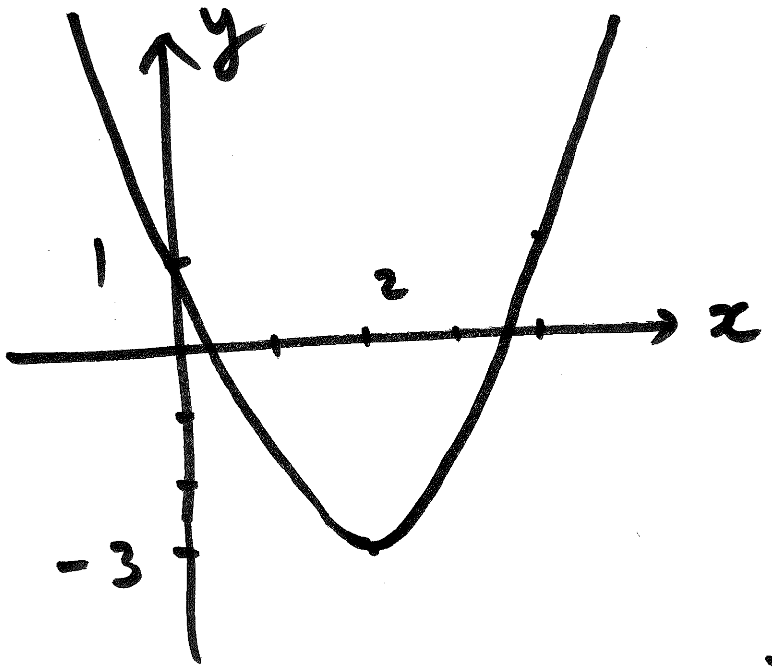
i.e. $5 - 2x_1 = 5 - 2x_2$. Simplify "5".

We get $-2x_1 = -2x_2$. Now cancel "-2"

$\Longrightarrow x_1 = x_2$.

Example 10:

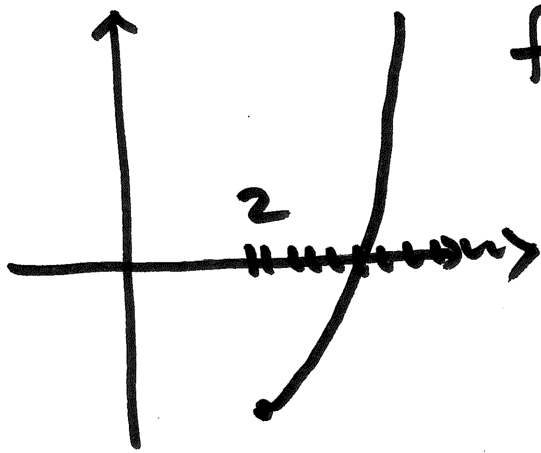
Graph the function $f(x) = (x-2)^2 - 3$. The function is not one-to-one: Why? Can you restrict its domain so that the resulting function is one-to-one? (There is more than one correct answer.)



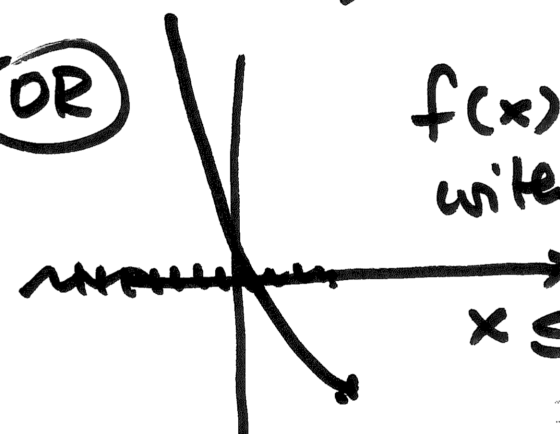
it is not one-to-one as it fails the horizontal line test.

To make it one-one

$f(x)$ with $x \geq 2$



OR



$f(x)$ with $x \leq 2$

The Inverse of a Function

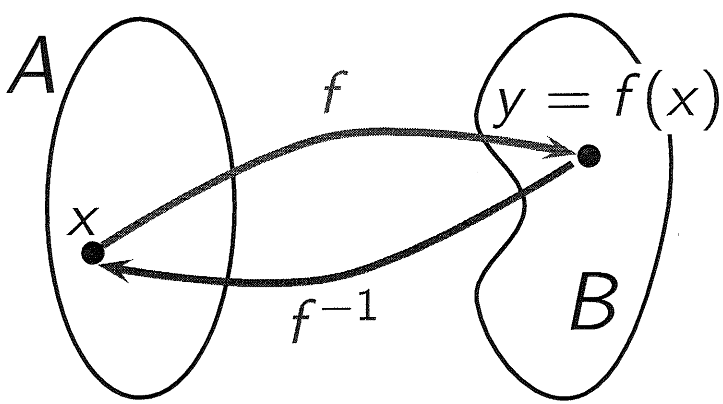
One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Its **inverse function** f^{-1} has domain B and range A and is defined by

$$\boxed{f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y,}$$

for any $y \in B$.



If f takes x to y,
then f^{-1} takes y back to x.
I.e., f^{-1} undoes what f does.

NOTE:
f^{-1} does NOT mean $\dfrac{1}{f}$.

Example 11:

Suppose $f(x)$ is a one-to-one function.

If $f(2) = 7$, $f(3) = -1$, $f(5) = 18$, $f^{-1}(2) = 6$ find:

$f^{-1}(7) = 2$ $\qquad$ $f(6) = 2$

$f^{-1}(-1) = 3$ $\qquad$ $f(f^{-1}(18)) = 18$

If $g(x) = 9 - 3x$, then $g^{-1}(3) = 2$

Suppose $9 - 3x = g(x) = 3$ then

$-3x = -6 \Longrightarrow \boxed{x = 2}$

Properties of Inverse Functions

Let $f(x)$ be a one-to-one function with domain A and range B. The inverse function $f^{-1}(x)$ satisfies the following "cancellation" properties:

$$\boxed{f^{-1}(f(x)) = x \quad \text{for every} \quad x \in A}$$

$$\boxed{f(f^{-1}(x)) = x \quad \text{for every} \quad x \in B}$$

Conversely, any function $f^{-1}(x)$ satisfying the above conditions is the inverse of $f(x)$.

Example 12:

Show that the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverses of each other.

$$f(g(x)) = [g(x)]^5 = [x^{1/5}]^5 = x$$

$$g(f(x)) = [f(x)]^{1/5} = [x^5]^{1/5} = x$$

Example 13:

Show that the functions $f(x) = \dfrac{1+3x}{5-2x}$ and $g(x) = \dfrac{5x-1}{2x+3}$ are inverses of each other.

we do one of the verifications : $f(g(x)) = x$ ....

$$f(g(x)) = \frac{1+3g(x)}{5-2g(x)} = \frac{1+3\left(\frac{5x-1}{2x+3}\right)}{5-2\left(\frac{5x-1}{2x+3}\right)} =$$

$$= \frac{\frac{(2x+3)+3(5x-1)}{2x+3}}{\frac{5(2x+3)-2(5x-1)}{2x+3}} = \frac{\cancel{17}x}{\cancel{2x+3}} \cdot \frac{\cancel{2x+3}}{\cancel{17}}$$

$$= \underline{x}$$

How to find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 14:

Find the inverse of $y = 4x - 7$.

1. $y = 4x - 7$

2. $4x = y + 7 \longrightarrow x = \frac{1}{4}y + \frac{7}{4}$

3. $\boxed{y = \frac{1}{4}x + \frac{7}{4}}$

Example 15:

Find the inverse of $y = \dfrac{1}{x+2}$.

1. $y = \dfrac{1}{x+2}$

2. $x + 2 = \dfrac{1}{y} \rightsquigarrow x = \dfrac{1}{y} - 2$

3. $\boxed{y = \dfrac{1}{x} - 2}$ or $\boxed{y = \dfrac{1-2x}{x}}$

Example 16:

Find the inverse of $y = \dfrac{2-x}{x+2}$.

① $y = \dfrac{2-x}{x+2}$

② $y(x+2) = 2-x \rightsquigarrow xy + x = 2-2y$

$\rightsquigarrow x(y+1) = 2-2y \longrightarrow x = \dfrac{2-2y}{y+1}$

③ $\boxed{y = \dfrac{2-2x}{x+1}}$

Graph of the Inverse Function

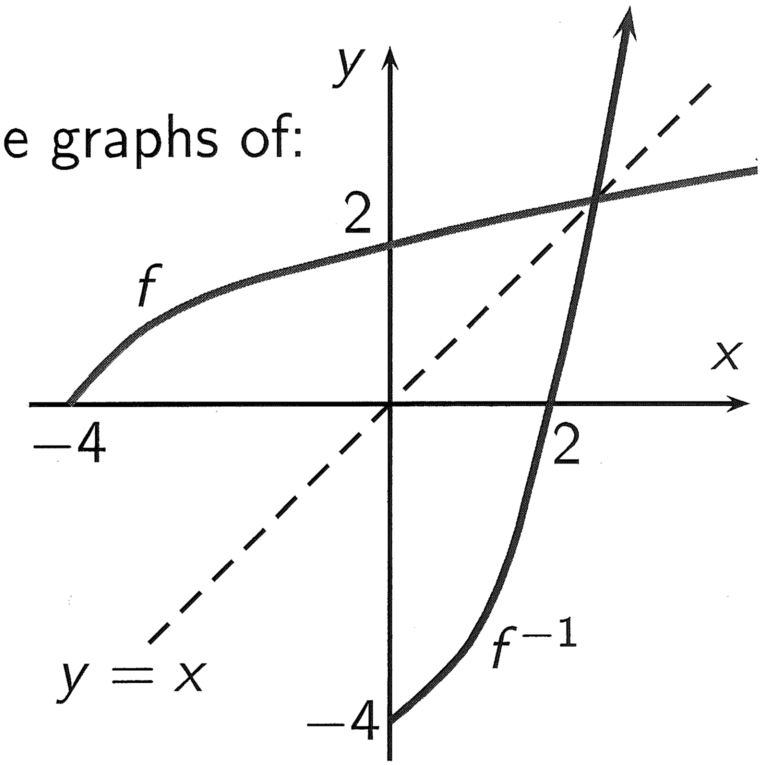
The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**

The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4}$$

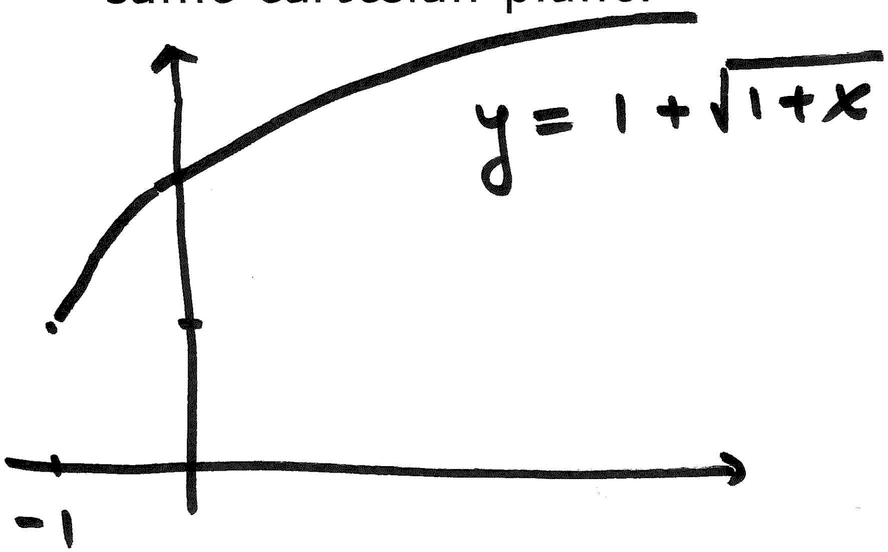
and

$$f^{-1}(x) = x^2 - 4, \; x \geq 0.$$



Example 17:

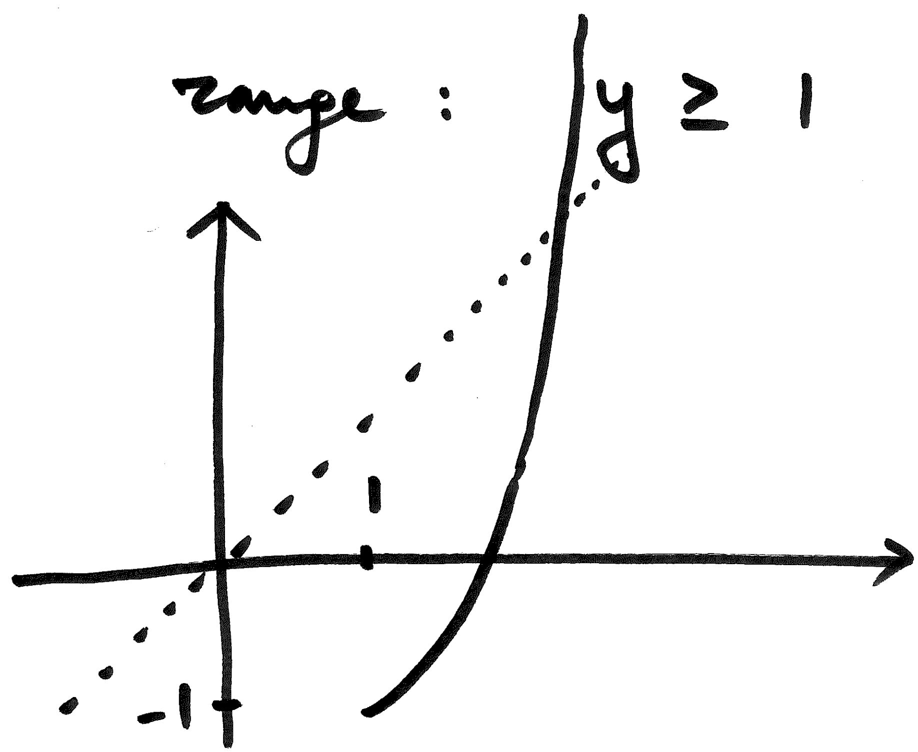
Find the inverse of the function $f(x) = 1 + \sqrt{1+x}$.
Find the domain and range of f and f^{-1}. Graph f and f^{-1} on the same cartesian plane.



domain: $x \geq -1$

range: $y \geq 1$

graph of inverse



the domain of the inverse is : $\boxed{x \geq 1}$

the range is : $y \geq -1$

To get the expression of the inverse

① $y = 1 + \sqrt{1+x}$

② $y-1 = \sqrt{1+x} \rightsquigarrow (y-1)^2 = (\sqrt{1+x})^2$

$y^2 - 2y + 1 = 1 + x \rightsquigarrow x = y^2 - 2y$

③ $\boxed{y = x^2 - 2x}$ with $x \geq 1$