FastTrack — MA 137/MA 113 — BioCalculus

Functions (4):
Exponential and Logarithmic Functions

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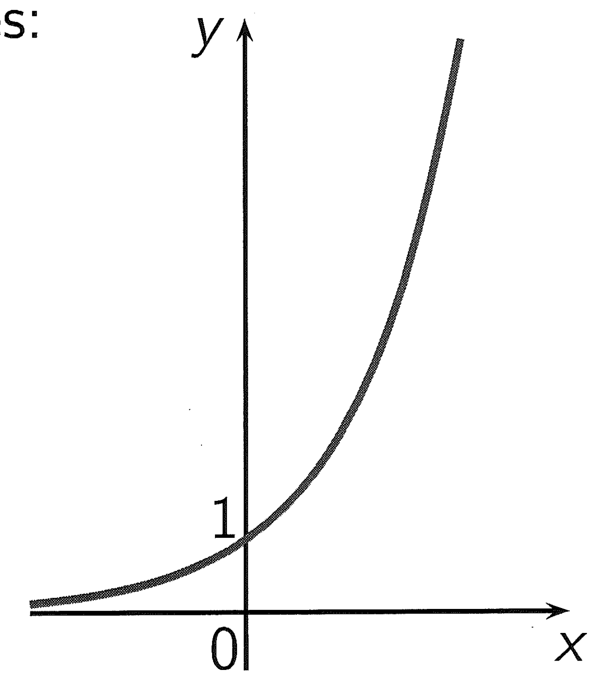
Goal: We introduce two new classes of functions called *exponential and logarithmic functions*. They are inverses of each other. Exponential functions are appropriate for modeling such natural processes as population growth for all living things and radioactive decay.

Exponential Functions

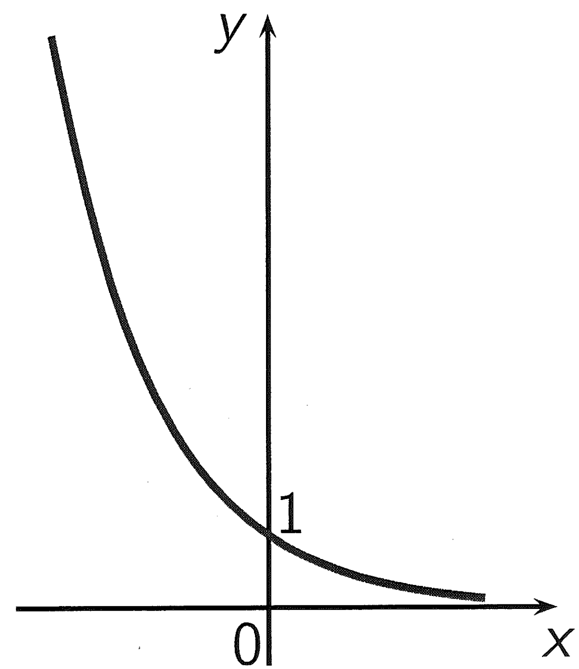
The **exponential function**

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$
for $0 < a < 1$

Example 1:

Let $f(x) = 2^x$. Evaluate the following:

$f(2) = 2^2 = \boxed{4}$

$f(-1/3) = 2^{-1/3} = \frac{1}{2^{1/3}} = \frac{1}{\sqrt[3]{2}}$

$\cong \boxed{0.793}$

$f(\pi) = 2^{\pi} \cong \boxed{8.825}$

$f(-\sqrt{3}) = 2^{-\sqrt{3}} = \frac{1}{2^{\sqrt{3}}}$

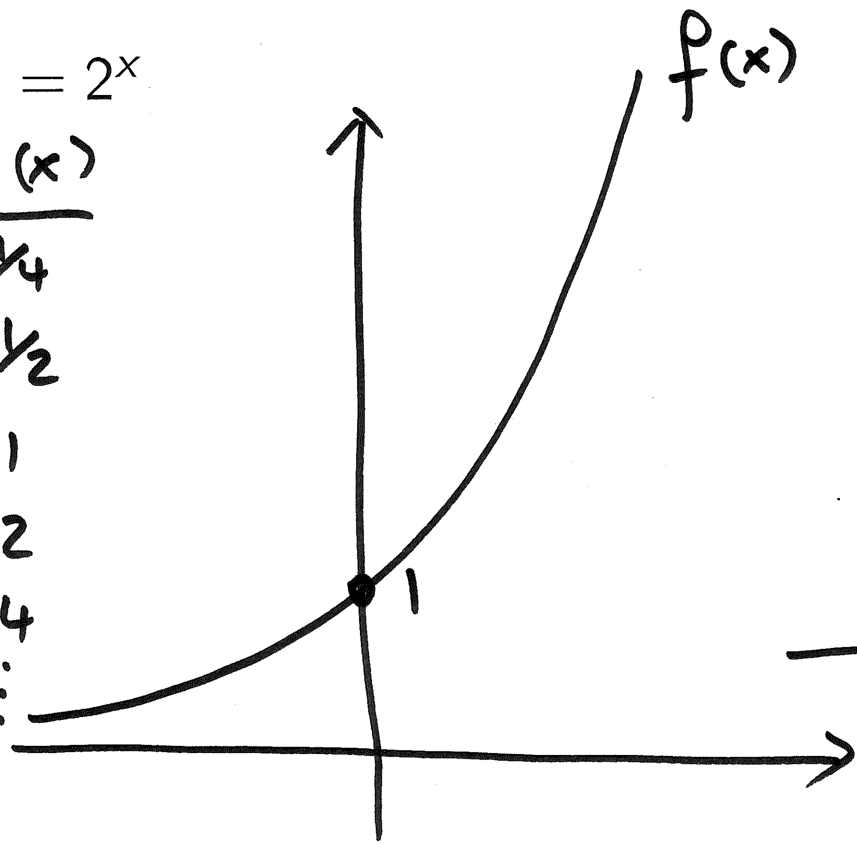
$\cong \boxed{0.301}$

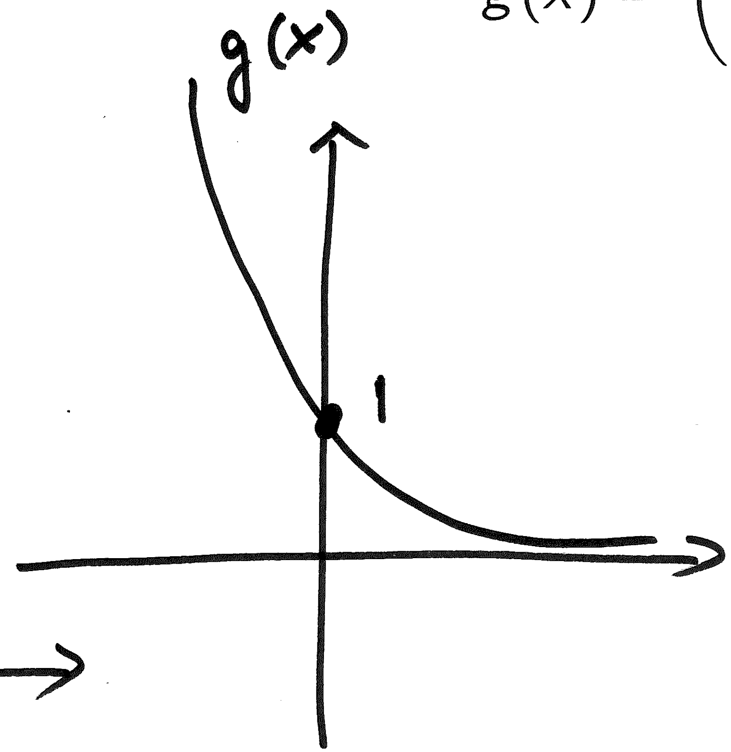
Example 2:

Draw the graph of each function:

$f(x) = 2^x$

x	$f(x)$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
$\vdots$	$\vdots$





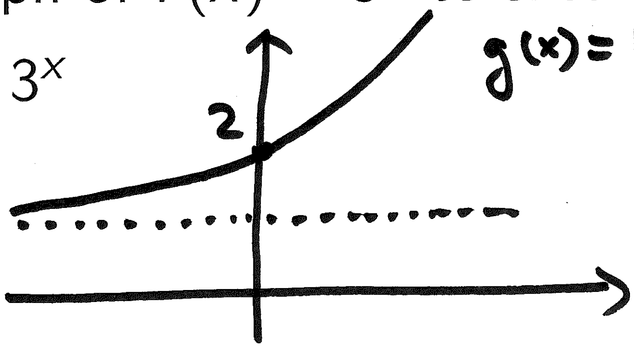
$g(x) = \left(\frac{1}{2}\right)^x$

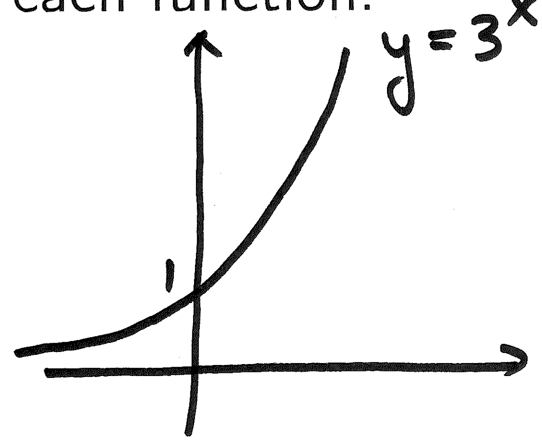
x	$g(x)$
-2	4
-1	2
0	1
1	$1/2$
2	$1/4$
$\vdots$	

Example 3:

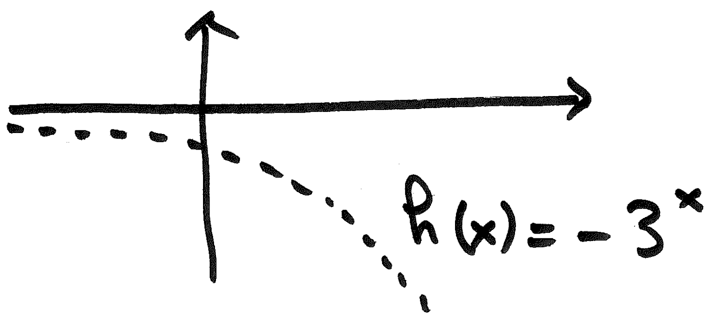
Use the graph of $f(x) = 3^x$ to sketch the graph of each function:

$g(x) = 1 + 3^x$

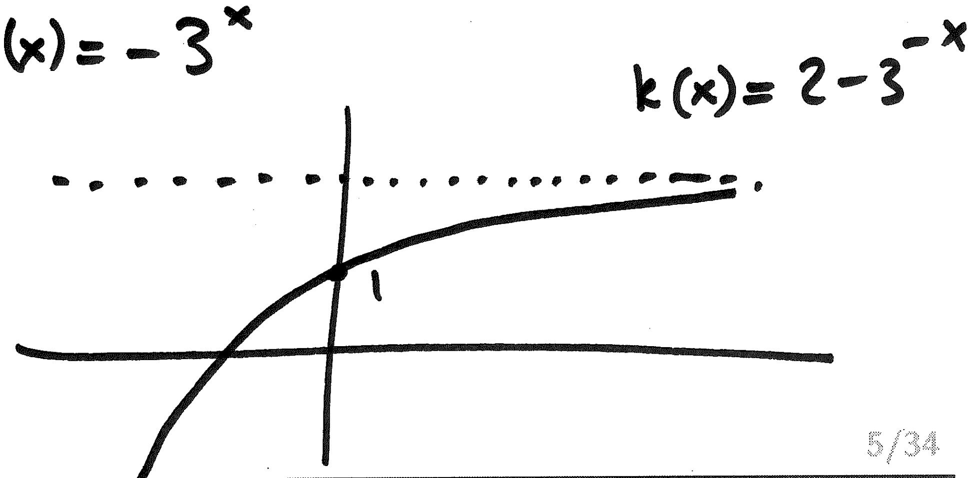




$h(x) = -3^x$



$k(x) = 2 - 3^{-x}$



The Number ‘e’

The most important base is the number denoted by the letter e.

The number e is defined as the value that $(1+1/n)^n$ approaches as n becomes very large.

Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

The Natural Exponential Function

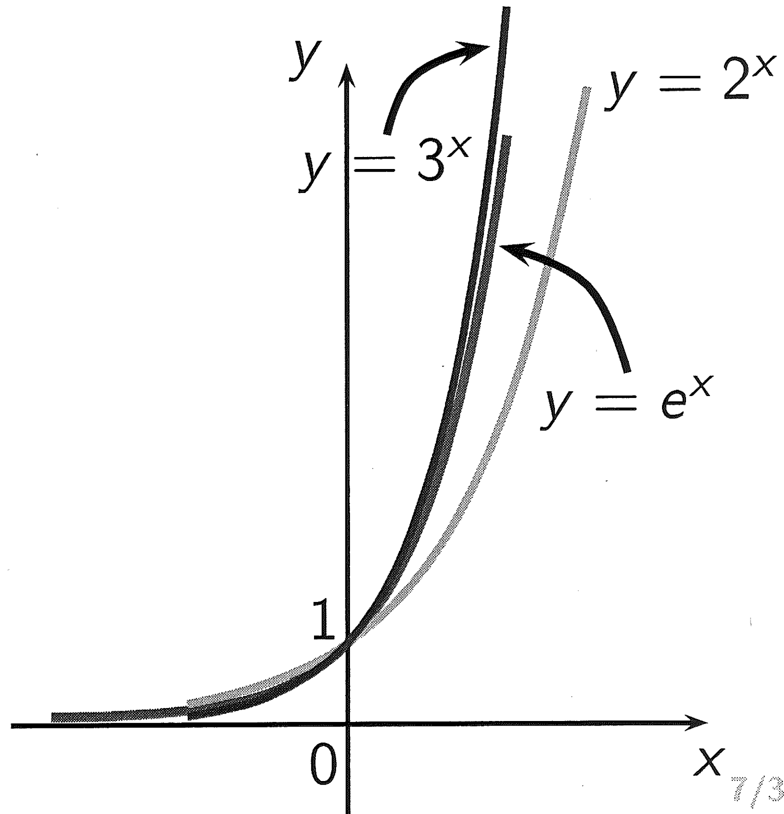
The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



Example 4:

When a certain drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50\, e^{-0.2t}.$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

$$D(3) = 50\, e^{-0.2\cdot 3} = 50\, e^{-0.6} \cong 27.44\,\text{mg}$$

Compound Interest

Compound interest is calculated by the formula:

$$\boxed{P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}}$$

where

$$\begin{aligned} P(t) &= \text{principal after } t \text{ years} \\ P_0 &= \text{initial principal} \\ r &= \text{interest rate per year} \\ n &= \text{number of times interest is compounded per year} \\ t &= \text{number of years} \end{aligned}$$

Continuously Compounded Interest

Continuously compounded interest is calculated by the formula:

$$\boxed{P(t) = P_0\, e^{rt}}$$

where

$P(t)$ = principal after t years
r = interest rate per year
P_0 = initial principal
t = number of years

Proof: The interest paid increases as the number n of compounding periods increases. If $m = \dfrac{n}{r}$, then:

$$P\left(1+\frac{r}{n}\right)^{nt} = P\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

But as m becomes large, the quantity $(1+1/m)^m$ approaches the number e. Thus, we obtain the formula for the continuously compounded interest.

Example 5:

Suppose you invest \$2,000 at an annual rate of 12% ($r = 0.12$) compounded quarterly ($n = 4$). How much money would you have one year later? What if the investment was compounded monthly ($n = 12$)?

$n=4$

$$A(t) = 2{,}000\left(1 + \frac{0.12}{4}\right)^{4t} = 2{,}000\,(1.03)^{4t}$$

So: $A(1) = 2{,}000\,(1.03)^4 \cong \$\,2{,}251.02$

$n=12$

$$A(t) = 2{,}000\left(1 + \frac{0.12}{12}\right)^{12t} = 2{,}000\,(1.01)^{12t}$$

So: $A(1) = 2{,}000\,(1.01)^{12} \cong \$\,2{,}253.65$

Example 6:

Suppose you invest $2,000 at an annual rate of 9% ($r = 0.09$) compounded continuously. How much money would you have after three years?

$$A(t) = 2{,}000\, e^{0.09t}$$

So:

$$A(3) = 2{,}000\, e^{0.09 \cdot 3}$$
$$= 2{,}000\, e^{0.27}$$
$$\cong \$\,2{,}619.93$$

Logarithmic Functions

Every exponential function $f(x) = a^x$, with $0 < a \neq 1$, is a one-to-one function (Horizontal Line Test). Thus, it has an inverse function, called the *logarithmic function with base a* and denoted by $\log_a x$.

Definition

Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a x \quad \Longleftrightarrow \quad a^y = x.$$

In other words, $\log_a x$ is the exponent to which a must be raised to give x.

Properties of Logarithms

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

Example 7:

Change each exponential expression into an equivalent expression in logarithmic form:

$5^3 = b \quad \longleftrightarrow \quad \log_5(b) = 3$

$a^6 = 15 \quad \longleftrightarrow \quad \log_a(15) = 6$

$e^{t+1} = 0.5 \quad \longleftrightarrow \quad \log_e(0.5) = t+1$

Example 8:

Change each logarithmic expression into an equivalent expression in exponential form:

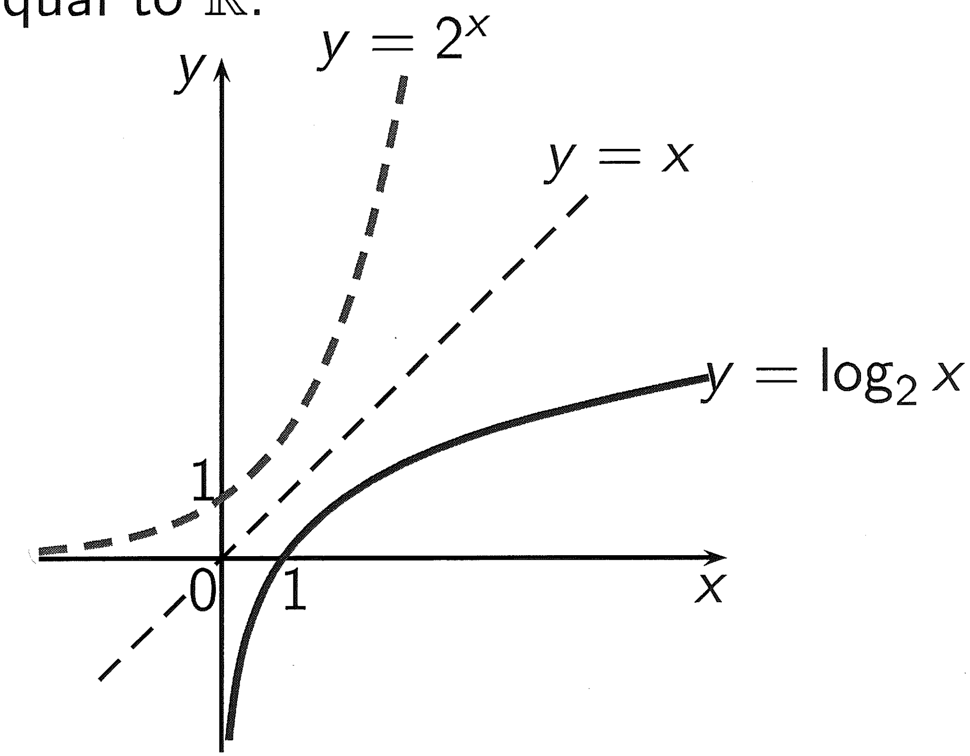
$\log_3 81 = 4 \quad \longleftrightarrow \quad 3^4 = 81$

$\log_8 4 = \dfrac{2}{3} \quad \longleftrightarrow \quad 8^{2/3} = 4$

$\log_e(x-3) = 2 \quad \longleftrightarrow \quad e^2 = x-3$

Graphs of Logarithmic Functions

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Thus, the function $y = \log_a x$ is defined for $x > 0$ and has range equal to $\mathbb{R}$.

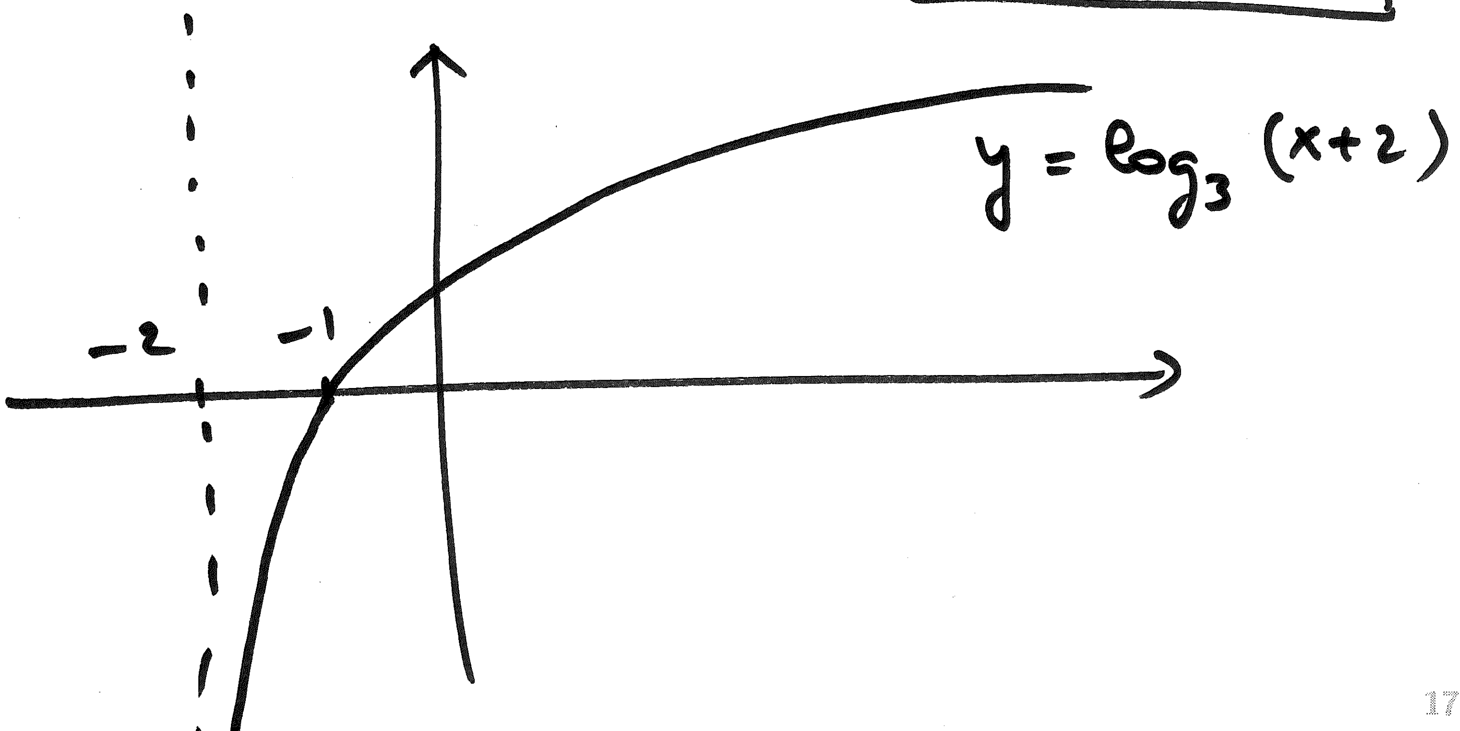


The point $(1, 0)$ is on the graph of $y = \log_a x$ (as $\log_a 1 = 0$) and the y-axis is a vertical asymptote.

Example 9:

Find the domain of the function $f(x) = \log_3(x+2)$ and sketch its graph.

domain : $x + 2 > 0 \iff \boxed{x > -2}$



Common Logarithms

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base: $\boxed{\log x := \log_{10} x.}$

Example 10 (Bacteria Colony):

A certain strain of bacteria divides every three hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\,\frac{\log(N/50)}{\log 2}.$$

Find the time required for the colony to grow to a million bacteria.

when $N = 1{,}000{,}000$ then

$$t = 3 \frac{\log\left(\frac{1{,}000{,}000}{50}\right)}{\log 2}$$

$$= 3 \frac{\log(20000)}{\log(2)} \cong \boxed{42.86 \text{ hours}}$$

Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition

The logarithm with base e is called the **natural logarithm** and denoted:

$$\boxed{\ln x := \log_e x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \quad \Longleftrightarrow \quad e^y = x.$$

Properties of Natural Logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Example 11:

Evaluate each of the following expressions:

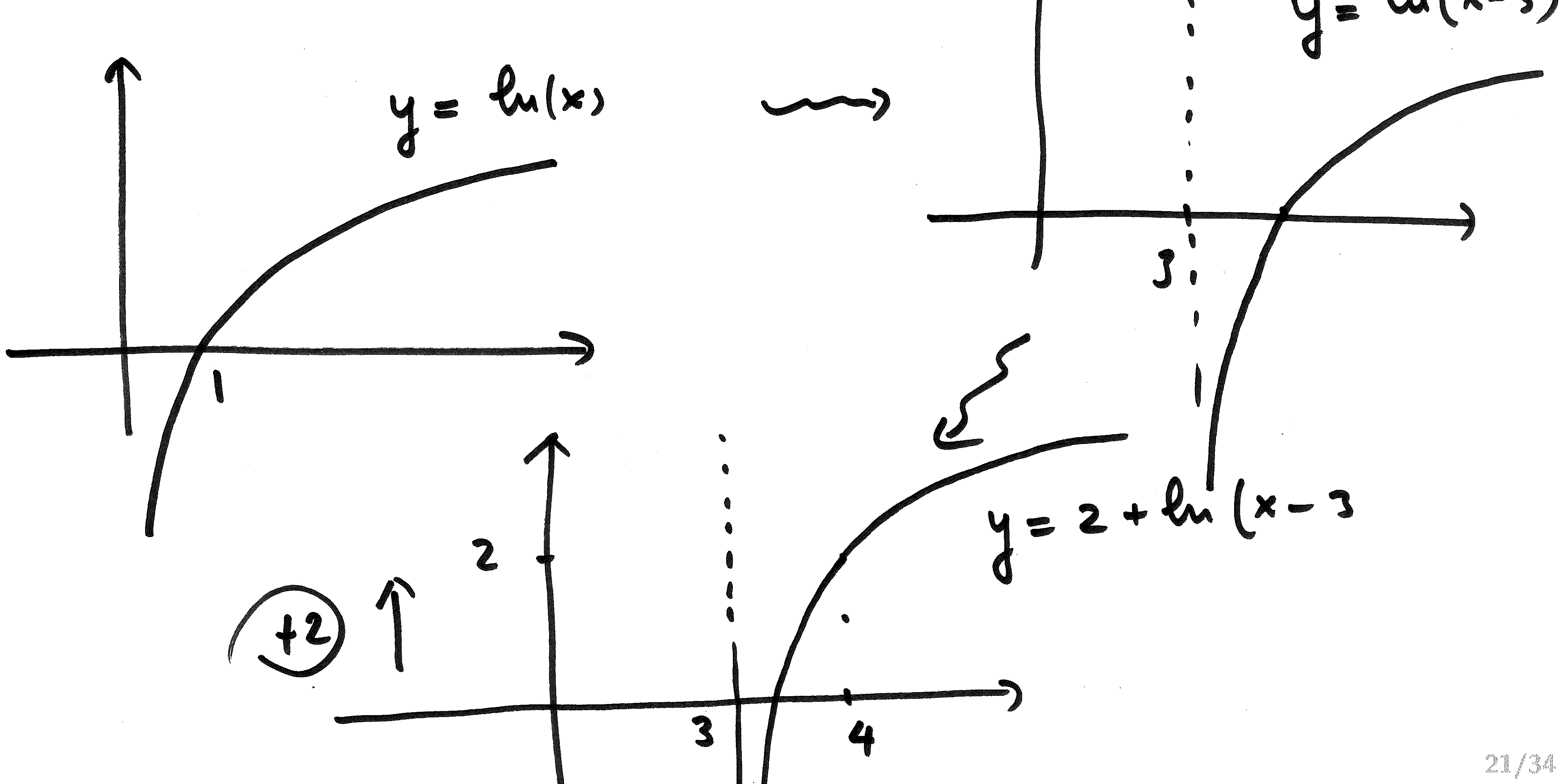
$\ln e^9 = \boxed{9}$

$\ln \frac{1}{e^4} = \boxed{\ln(e^{-4}) = -4}$

$e^{\ln 2} = \boxed{2}$

Example 12:

Graph the function $y = 2 + \ln(x - 3)$.



Example 13:

Find the domain of the function $f(x) = 2 + \ln(10 + 3x - x^2)$.

$f(x)$ is defined when $10 + 3x - x^2 > 0$

$\Longleftrightarrow \quad x^2 - 3x - 10 < 0 \quad \Longleftrightarrow \quad (x-5)(x+2) < 0$

$x - 5$: $- - - - - - -$ 5 $+ + +$

$x + 2$: $- - - - -$ -2 $+ + + + + + + + + +$

$(x-5)(x+2)$: $+ +$ -2 $- - - -$ 5 $+ + + +$

domain:

$$\boxed{-2 < x < 5}$$

Laws of Logarithms

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

1. $\log_a(AB) = \log_a A + \log_a B$;
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$;
3. $\log_a(A^C) = C \log_a A$.

Proof of **Law 1.**: $\log_a(AB) = \log_a A + \log_a B$

Let us set

$$\log_a A = u \quad \text{and} \quad \log_a B = v.$$

When written in exponential form, they become

$$a^u = A \quad \text{and} \quad a^v = B.$$

Thus:

$$\begin{aligned} \underline{\log_a(AB)} &= \log_a(a^u a^v) \\ &= \log_a(a^{u+v}) \\ &\overset{\text{why?}}{=} u + v \\ &= \underline{\log_a A + \log_a B}. \end{aligned}$$

In a similar fashion, one can prove **2.** and **3.**

Example 14:

Evaluate each expression:

$\log_5 5^9$

$= 9$

$\log_3 7 + \log_3 2$

$= \log_3(14)$

$\log_3 16 - 2\log_3 2$

$= \log_3 16 - \log_3 2^2$

$= \log_3\left(\frac{16}{4}\right) = \log_3 4$

$\ln\left(\ln e^{(e^{200})}\right)$

$= \ln\left[e^{200} \cdot \underbrace{\ln e}_{1}\right]$

$= \ln(e^{200}) = 200 \underbrace{\ln e}_{1} = 200$

$\log_3 100 - \log_3 18 - \log_3 50$

$= \log_3\left(\frac{100}{18 \cdot 50}\right)$

$= \log_3(1/9) =$

$= \log_3(3^{-2}) = -2$

Expanding and Combining Logarithmic Expressions

Example 15:

Use the Laws of Logarithms to expand each expression:

$\log_2(2x) = \log_2(2) + \log_2(x) = \boxed{1 + \log_2 x}$

$\log_5(x^2(4-5x)) = \log_5 x^2 + \log_5(4-5x)$

$= \boxed{2\log_5 x + \log_5(4-5x)}$

$\log\left(x\sqrt{\dfrac{y}{z}}\right) = \log x + \log\left[\left(\dfrac{y}{z}\right)^{1/2}\right] =$

$= \log x + \dfrac{1}{2}\left[\log\left(\dfrac{y}{z}\right)\right] = \boxed{\log x + \dfrac{1}{2}\log y - \dfrac{1}{2}\log z}$

Example 16:

Use the Laws of Logarithms to combine the expression

$$\log_a b + c \log_a d - r \log_a s$$

into a single logarithm.

$$\log_a b + \log_a d^c - \log_a s^r$$

$$= \boxed{\log_a \left(\frac{b d^c}{s^r} \right)}$$

Example 17:

Use the Laws of Logarithms to combine the expression

$$\ln 5 + \ln(x+1) + \frac{1}{2}\ln(2-5x) - 3\ln(x-4) - \ln x$$

into a single logarithm.

$$= \ln 5 + \ln(x+1) + \ln\sqrt{2-5x} - \left[\ln(x-4)^3 + \ln x\right]$$

$$= \boxed{\ln\left[\frac{5(x+1)\sqrt{2-5x}}{(x-4)^3 \cdot x}\right]}$$

Example 18 (Forgetting):

Ebbinghaus's Law of Forgetting states that if a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1),$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve the equation for P.

(b) Use Ebbinghaus's Law of Forgetting to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume $c = 0.3$.

(a) $$\log P = \log P_0 - \log\left[(t+1)^c\right]$$

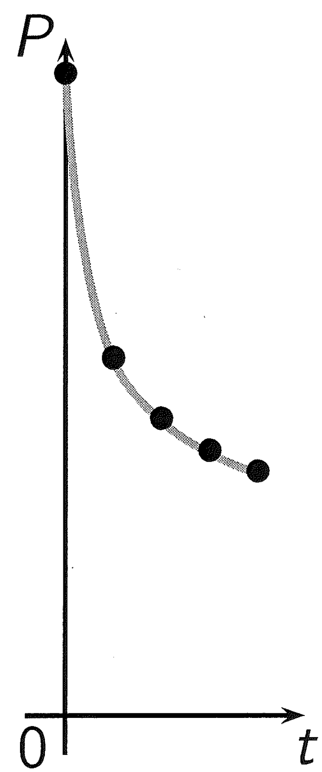
$$\therefore \quad \log P = \log\left[\frac{P_0}{(t+1)^c}\right]$$

$$\therefore \quad P = \frac{P_0}{(t+1)^c} = \boxed{P_0\,(t+1)^{-c}} \quad /4$$

(b) $$P(\underbrace{24}_{\text{2 years in months}}) = \frac{80}{(24+1)^{0.3}} \cong \underline{\underline{30.46}}$$

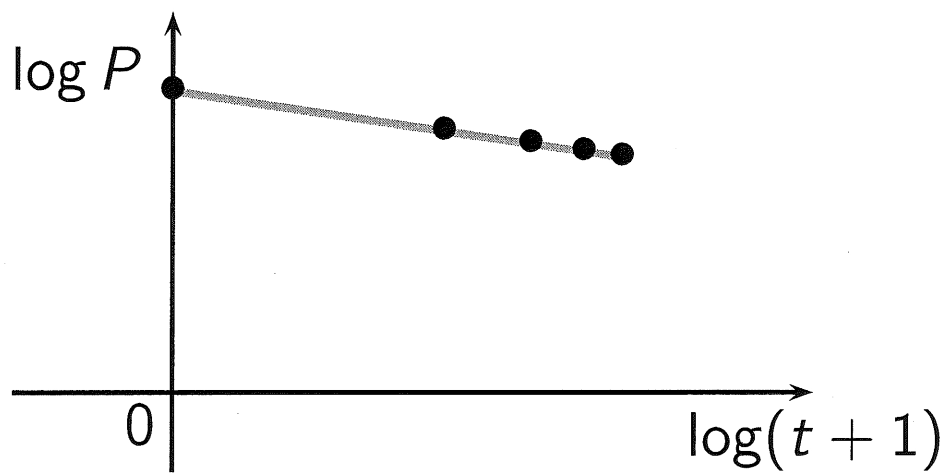
Comment (about Example 18)

t	$P = 80/(t+1)^{0.3}$
0	80
6	44.62
12	37.06
18	33.072
24	30.458



Comment (cont.d)

t	$\log(t+1)$	$\log P = \log 80 - 0.3\log(t+1)$
0	0	1.903
6	0.845	1.650
12	1.114	1.569
18	1.279	1.519
24	1.398	1.484



Example 19 (Biodiversity):

Some biologists model the number of species S in a fixed area A (such as an island) by the **Species-Area relationship**

$$\log S = \log c + k \log A,$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part **(a)** to show that if $k = 3$ then doubling the area increases the number of species eightfold.

(a) $\log(S) = \log(c) + \log(A^k)$

$= \log(cA^k)$

<u>So</u>: $\boxed{S = cA^k}$

(b) Suppose $S = cA^3$. Now suppose that for a certain value A_0 we obtain $S_0 = cA_0^3$. If we plug in into the formula $A_1 = 2A_0$ we get

$$S_1 = cA_1^3 = c(2A_0)^3 = 8\underbrace{cA_0^3} = \underline{8S_0}$$

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}.}$$

Proof: Set $y = \log_b x$. By definition, this means that $b^y = x$. Apply now $\log_a(\cdot)$ to $b^y = x$. We obtain

$$\log_a(b^y) = \log_a x \qquad \rightsquigarrow \qquad y \log_a b = \log_a x.$$

Thus

$$\log_b x = y = \frac{\log_a x}{\log_a b}.$$

Example 20:

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct up to five decimal places:

$\log_5 2 = \dfrac{\log 2}{\log 5} \cong 0.43068$

$\log_4 125 = \dfrac{\log 125}{\log 4} \cong 3.48289$

$\log_{\sqrt{3}} 5 = \dfrac{\log 5}{\log(\sqrt{3})} = \dfrac{\log 5}{\frac{1}{2}\log 3} \cong 2.92995$