FastTrack — MA 137/MA 113 — BioCalculus

Functions (5): Modeling with Exponential and Logarithmic Functions

Alberto Corso – ⟨alberto.corso@uky.edu⟩

Department of Mathematics – University of Kentucky

Goal: Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, can be modeled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena.

Exponential Equations

An exponential equation is one in which the variable occurs in the exponent. For example,

$$3^{x+2} = 7.$$

We take the (either common or natural) logarithm of each side and then use the Laws of Logarithms to 'bring down the variable' from the exponent:

$$\log(3^{x+2}) = \log 7$$

$$\begin{aligned} \rightsquigarrow \quad & (x+2)\log 3 = \log 7 \\ \rightsquigarrow \quad & x + 2 = \frac{\log 7}{\log 3} \\ \rightsquigarrow \quad & x = \frac{\log 7}{\log 3} - 2 \approx -0.228756 \end{aligned}$$

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to 'bring down the exponent.'
3. Solve for the variable.
4. Check your answer.

Example 1:

Solve the equations:

- $3 \cdot 4^x = 18$ $\Longrightarrow$ $4^x = \frac{18}{3}$ $\rightsquigarrow$ $4^x = 6$

Take log of both sides: $\log(4^x) = \log 6$

$\longrightarrow$ $x \cdot \log 4 = \log 6$ $\Longrightarrow$ $x = \frac{\log 6}{\log 4} \approx \underline{1.2925}$

- $3^{x+4} = 2^{1-2x}$

$\log(3^{x+4}) = \log(2^{1-2x}) \rightsquigarrow (x+4)\log 3 = (1-2x)\log 2$

$\longrightarrow$ $x \log 3 + 2x \log 2 = \log 2 - 4 \log 3$

$\longrightarrow$ $x(\log 3 + 2\log 2) = \log\left(\frac{2}{3^4}\right)$ $\therefore$ $x = \frac{\log(2/3^4)}{\log(3 \cdot 2^2)} \cong \underline{-1.48}$

Example 2:

Solve the following exponential equations of quadratic type:

- $9^x - 3^x = 72$

$(3^2)^x - 3^x - 72 = 0$ or $(3^x)^2 - 3^x - 72 = 0$. Let $u = 3^x$ so that eq. becomes $u^2 - u - 72 = 0 \leadsto (u-9)(u+8) = 0$ $\therefore$ $u = 9$ or $u = -8$

Hence $u = 3^x = 9 \Longrightarrow \boxed{x = 2}$; $u = 3^x = -8$ imposs

- $4^x - 3(4^{-x}) = 2$

$4^x - \frac{3}{4^x} = 2 \leadsto (4^x)^2 - 3 = 2 \cdot 4^x \leadsto$

$(4^x)^2 - 2 \cdot 4^x - 3 = 0$. As before $u = 4^x \leadsto$

$u^2 - 2u - 3 = 0 \Longrightarrow (u-3)(u+1) = 0 \Longrightarrow u = 3$ or $u = -1$

$u = 4^x = 3 \Longrightarrow \boxed{x = \frac{\log 3}{\log 4}}$; $u = 4^x = -1$ impossible

Example 3:

Solve the equation $x^2e^x + xe^x - 6e^x = 0.$

factor as $e^x(x^2+x-6)=0$

$\Rightarrow$ $e^x = 0$ (which is impossible) ✗

or $x^2+x-6=0 \Rightarrow (x+3)(x-2)=0$

$\therefore$ $\underline{x=-3, 2}$

Logarithmic Equations

A logarithmic equation is one in which a logarithm of the variable occurs. For example,

$$\log_2(25 - x) = 3.$$

To solve for x, we write the equation in exponential form, and then solve for the variable:

$$25 - x = 2^3 \quad \rightsquigarrow \quad 25 - x = 8 \quad \rightsquigarrow \quad x = 17.$$

Alternatively, we raise the base, 2, to each side of the equation; we then use the Laws of Logarithms:

$$2^{\log_2(25-x)} = 2^3 \quad \rightsquigarrow \quad 25 - x = 2^3 \quad \rightsquigarrow \quad x = 17.$$

Guidelines for Solving Logarithmic Equations

1. Isolate the logarithmic term on one side of the equation; you may first need to combine the logarithmic terms.

2. Write the equation in exponential form (or raise the base to each side of the equation).

3. Solve for the variable. Check your answers!

Example 4:

Solve the following equations:

- $2\log_7 x = \log_7 16$ $\Longrightarrow$ $\log_7 x^2 = \log_7 16$

$\Longrightarrow 7^{\log_7 x^2} = 7^{\log_7 16} \Longrightarrow x^2 = 16 \quad \therefore x = 4, \cancel{-4}$

- $\log_2(x+3) = \log_2(x-3) + \log_3 9 + 4^{\log_4 3}$

$\log_2(x+3) - \log_2(x-3) = 2 + 3$

$\Longrightarrow \log_2\left[\frac{x+3}{x-3}\right] = 5$ or $2^5 = \frac{x+3}{x-3} \Longrightarrow$

$32(x-3) = x+3 \Longrightarrow 32x - x = 3 + 96 \quad \therefore x = \frac{99}{31}$

Example 5:

Solve the following equations:

- $\log_6(x+5) + \log_6 x = 2$

$\log_6[(x+5)\cdot x] = 2 \implies 6^2 = x^2 + 5x$

$\implies x^2 + 5x - 36 = 0 \implies (x+9)(x-4) = 0$

$\therefore\ x = \cancel{-9},\ 4$

- $\log(x^3) = (\log x)^3$

$3 \log x = (\log x)^3$. Set $u = \log x$ and get

$3u = u^3 \implies u^3 - 3u = 0 \quad u(u^2 - 3) = 0$

$\implies u = 0, \pm\sqrt{3}$ since $\log x = u = 0, \pm\sqrt{3}$

$\implies x = 1,\ 10^{\sqrt{3}},\ 10^{-\sqrt{3}}$

Example 6 (Genetic Mutation):

The basic source of genetic diversity is mutation (that is, changes in the chemical structure of genes). If genes mutate at a constant rate m (with $0 < m < 1$) and if other evolutionary forces are neglegible, then the frequency F of the original gene after t generations is

$$F = F_0(1-m)^t,$$

where F_0 is the frequency at $t = 0$.

(a) Solve the above equations for t, using log.
(b) If $m = 5 \times 10^{-5}$, after how many generations is $F/F_0 = 1/2$?

(a) $F = F_0 (1-m)^t \implies F/F_0 = (1-m)^t$

$$\implies \log(F/F_0) = \log\left[(1-m)^t\right] = t \cdot \log(1-m)$$

$$\therefore \boxed{t = \frac{\log(F/F_0)}{\log(1-m)}}$$

(b) $$t = \frac{\log(1/2)}{\log(1-5\cdot 10^{-5})} \approx 13{,}862.6$$

<u>generations</u>

Example 7 (Frog Population):

The frog population in a small pond grows exponentially. The current population is 85 frogs, and the relative growth rate is 18% per year.

(a) Which function models the population after t years?
(b) Find the projected frog population after 3 years.
(c) When will the frog population reach 600?
(d) When will the frog population double?

(a) $n(t) = 85 \cdot e^{0.18t}$

(b) $n(3) = 85 \cdot e^{0.18 \cdot 3} = 85 \cdot e^{0.54}$

≈ 145.86

(c) $600 = 85 \cdot e^{0.18\bar{t}} \Longrightarrow$

$\frac{600}{85} = e^{0.18\bar{t}} \Rightarrow \ln\left(\frac{600}{85}\right) = \ln\left(e^{0.18\bar{t}}\right)$

$\Rightarrow \ln\left(\frac{600}{85}\right) = 0.18\bar{t} \Rightarrow \bar{t} = \frac{\ln(600/85)}{0.18}$

$\Rightarrow \bar{t} \approx 10.857$ years

(d) Let h be the doubling time:

$$\cancel{85} \cdot 2 = n(h) = \cancel{85} \cdot e^{0.18 h}$$

$$\Rightarrow \quad 2 = e^{0.18 \cdot h}$$

$$\Rightarrow \quad \ln(2) = \ln\left(e^{0.18 \cdot h}\right) \overset{!}{=} 0.18 \cdot h$$

$$\therefore \quad h = \frac{\ln 2}{0.18} \approx 3.85 \text{ years}$$

Exponential Models of Population Growth

The formula for population growth of several species is the same as that for continuously compounded interest. In fact in both cases the rate of growth r of a population (or an investment) per time period is proportional to the size of the population (or the amount of an investment).

Exponential Growth Model

If n_0 is the initial size of a population that experiences **exponential growth**, then the population $n(t)$ at time t increases according to the model

$$\boxed{n(t) = n_0 e^{rt}}$$

where r is the relative rate of growth of the population (expressed as a proportion of the population).

Remark:

Biologists sometimes express the growth rate in terms of the **doubling-time** h, the time required for the population to double in size: $r = \dfrac{\ln 2}{h}$.

Proof: Indeed, from

$$2n_0 = n(h) = n_0 e^{rh}$$

we obtain

$$2 = e^{rh} \quad \rightsquigarrow \quad \ln 2 = rh \quad \rightsquigarrow \quad r = \frac{\ln 2}{h}.$$

Using the doubling-time h, we can also rewrite $n(t)$ as:

$$\boxed{n(t)} = n_0 e^{rt} = n_0 e^{\frac{\ln(2)}{h} t} = n_0 e^{\ln(2^{t/h})} \boxed{= n_0 2^{t/h}}.$$

Example 8 (Bacteria Culture):

The initial count in a culture of bacteria (growing exponentially) was 50. The count was 400 after 2 hours.

(a) What is the relative rate of growth of the bacteria population?
(b) When will the number of bacteria be 50,000?

(a) $n(t) = 50e^{rt}$. We know that

$400 = n(2) = 50e^{r\cdot 2} \Longrightarrow 8 = e^{2r}$

$\Longrightarrow \ln 8 = \ln(e^{2r}) \quad \therefore \quad r = \frac{\ln 8}{2} \approx 1.0397$

(or $r \approx 103.97\%$)

(b) $n(t) = 50e^{1.0397t}$ so that $50{,}000 = 50e^{1.0397\bar{t}}$

$\Longrightarrow 1{,}000 = e^{1.0397\bar{t}} \Longrightarrow \bar{t} = \frac{\ln(1000)}{1.0397} \approx 6.6$ hours

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiations. Also in this situation, the rate of decay is proportional to the mass of the substance.

This is analogous to population growth, except that the mass of radioactive material *decreases*.

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass $m(t)$ remaining at time t is modeled by the function

$$\boxed{m(t) = m_0 e^{-rt}}$$

where r is the relative rate of decay of the radioactive substance.

Remark:

Physicists sometimes express the rate of decay in terms of the **half-life** h, the time required for half the mass to decay: $r = \dfrac{\ln 2}{h}$.

Proof: Indeed, from

$$\frac{1}{2}m_0 = m(h) = m_0 e^{-rh}$$

we obtain

$$\frac{1}{2} = e^{-rh} \quad \rightsquigarrow \quad \ln\frac{1}{2} = -rh \quad \rightsquigarrow \quad -\ln 2 = -rh \quad \rightsquigarrow \quad r = \frac{\ln 2}{h}.$$

Using the half-time h, we can also rewrite $m(t)$ as:

$$\boxed{m(t)} = m_0 e^{-rt} = m_0 e^{-\frac{\ln(2)}{h}t} = m_0 e^{\ln(2^{-t/h})} \boxed{= m_0 \left(\frac{1}{2}\right)^{t/h}}.$$

Example 9:

The mass $m(t)$ remaining after t days from a 40-g sample of thorium-234 is given by:

$$m(t) = 40e^{-0.0277\,t}.$$

(a) How much of the sample will be left after 60 days?

(b) After how long will only 10-g of the sample remain?

(a) $m(60) = 40 \cdot e^{-0.0277 \cdot 60} \approx \underline{7.59036 \text{ grams}}$

(b) $10 = 40 \cdot e^{-0.0277 \cdot \bar{t}} \Longrightarrow$

$\frac{1}{4} = e^{-0.0277\bar{t}} \Rightarrow \ln(1/4) = \ln e^{-0.0277\bar{t}}$

$\Rightarrow \bar{t} = \frac{\ln 4}{0.0277} \approx \underline{50.046 \text{ days}}$

Example 10:

The half-life of cesium-137 is 30 years. Suppose we have a 10-g sample. How much of the sample will remain after 80 years?

We have $h = \text{half-life} = \frac{\ln 2}{r}$ or

$$r = \frac{\ln 2}{h} = \frac{\ln 2}{30} \approx 0.0231 .$$

Thus $m(t) = 10e^{-0.231t}$.

Finally $m(80) = 10 \cdot e^{-0.231 \cdot 80} \approx \underline{\underline{1.575 g}}$.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. Using Calculus, the following model can be deduced from this law:

The Model

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_S, then the temperature of the object at time t is modeled by the function

$$\boxed{T(t) = T_S + D_0 e^{-kt}}$$

where k is a positive constant that depends on the object.

Example 11 (Cooling Turkey):

A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 minutes?

(b) When will the turkey cool to 100°F?

(a) $D_0 = 185 - 75 = 110$ hence

$$T(t) = 75 + 110e^{-kt}$$

So $T(30) = 75 + 110e^{-k \cdot 30} = 150$

$$\Longrightarrow e^{-30k} = \frac{(150-75)}{110} = \frac{75}{110}$$

$$\Longrightarrow k = \frac{\ln(75/110)}{-30} \approx 0.01276$$

Hence $\boxed{T(t) = 75 + 110\, e^{-0.01276\, t}}$

and $T(45) \cong 136.93\ °F$

(b) Check that $T(\bar{t}) = 100\,°F$ for $\bar{t} = 116$ min
(almost 2 hours)

Remark:

Newton's Law of Cooling is used in homicide investigations to determine the time of death. Immediately following death, the body begins to cool (its normal temperature is $98.6°$F). It has been experimentally determined that the constant in Newton's Law of Cooling is $k \approx 0.1947$, assuming time is measured in hours.

Logarithmic Scales

When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers. We discuss the case of the **pH scale**, which measures acidity. You should refer to our textbook (Section 1.3) for other quantities that are measured on logarithmic scales; they include earthquake intensity (Richter scale), loudness of sounds (decibel scale), light intensity, information capacity, radiation, etc.

The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Sorensen, in 1909, defined a more convenient measure:

$$\boxed{\text{pH} = -\log[H^+]}$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter (M).

Solutions are defined in terms of the pH as follows:

those with $\text{pH} = 7$ (or $[H^+] = 10^{-7}\text{M}$) are *neutral*,

those with $\text{pH} < 7$ (or $[H^+] > 10^{-7}\text{M}$) are *acidic*,

those with $\text{pH} > 7$ (or $[H^+] < 10^{-7}\text{M}$) are *basic*.

Example 12 (Finding pH):

The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.

(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}\text{M}$

$$pH = -\log(5.0 \cdot 10^{-3}) = -\log(5) - \log 10^{-3} =$$
$$= 3 - \log 5 = \underline{2.301}$$

(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}\text{M}$

$$pH = -\log(3.2 \cdot 10^{-4}) = -\log 3.2 - \log 10^{-4}$$
$$= 4 - \log 3.2 = \underline{3.49485}$$

(c) Seawater: $[H^+] = 5.0 \times 10^{-9}\text{M}$

$$pH = -\log(5 \cdot 10^{-9}) = -\log(5) - \log(10^{-9}) =$$
$$= 9 - \log 5 = \underline{8.301}$$

Example 13 (Ion Concentration):

Calculate the hydrogen ion concentration of each substance from its pH reading.

(a) Vinegar: pH = 3.0

$$3.0 = -\log[H^+] \implies \log[H^+] = -3$$

$$\implies \underline{[H^+] = 10^{-3}}$$

(b) Milk: pH = 6.5

$$6.5 = -\log[H^+] \implies \log[H^+] = -6.5$$

$$\implies \underline{[H^+] = 10^{-6.5}}$$