FastTrack — MA 137/MA 113 — BioCalculus

Functions (1): Definitions and Basic Functions

Alberto Corso – ⟨alberto.corso@uky.edu⟩

Department of Mathematics – University of Kentucky

Goal: `Perhaps the most useful mathematical idea for modeling the real world is the concept of a` *function*. `We explore the idea of a function and then give its mathematical definition.`

Functions Around Us/Ways to Represent a Function

In nearly every physical phenomenon we observe that one quantity depends on another. For instance

- height is a function of age;
- temperature is a function of date;
- cost of mailing a package is a function of weight;
- the area of a circle is a function of its radius;
- the number of bacteria in a culture is a function of time;
- the price of a commodity is a function of the demand.

We can describe a specific function in the following four ways:

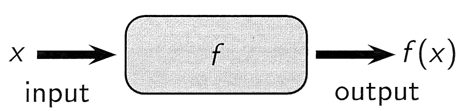
1. verbally (by a description in words);
2. algebraically (by an explicit formula);
3. visually (by a graph);
4. numerically (by a table of values).

Definition of Function

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

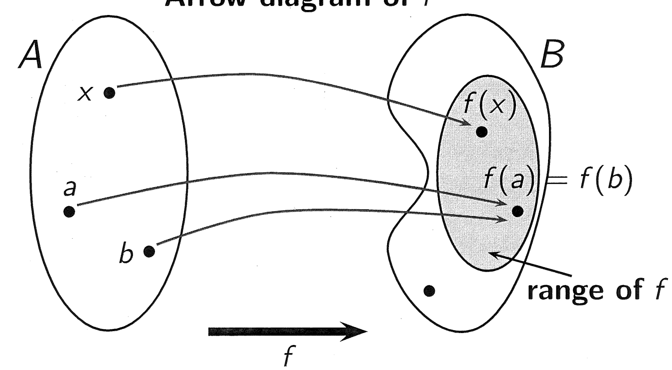
The set A is called the **domain** of f whereas the set B is called the **codomain** of f; $f(x)$ is called the **value of** f **at** x, or the **image of** x **under** f.

The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain: range of $f = \{f(x) \,|\, x \in A\}$.



Machine diagram of f

Arrow diagram of f



Notation: To define a function, we often use the notation

$$f : A \longrightarrow B, \qquad x \mapsto f(x)$$

where A and B are subsets of the set of real numbers $\mathbb{R}$.

Evaluating a Function:

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**.

The symbol that represents a number in the range of f is called a **dependent variable**.

In the definition of a function the independent variable plays the role of a "placeholder".

For example, the function $f(x) = 2x^2 - 3x + 1$ can be thought of as

$$f(\square) = 2 \cdot \square^2 - 3 \cdot \square + 1.$$

To evaluate f at a number (expression), we substitute the number (expression) for the placeholder.

Example 1:

Evaluate the piecewise function $f(x) = \begin{cases} x^2 + 2x & \text{if } x \leq -1 \\ x & \text{if } x > -1 \end{cases}$

at the indicated values:

$f(-4) = (-4)^2 + 2(-4) = 16 - 8 = \boxed{8}$

$f(-1) = (-1)^2 + 2(-1) = 1 - 2 = \boxed{-1}$

$f(0) = \boxed{0}$

$f(1) = \boxed{1}$

Example 2:

If $f(x) = 3 - 5x + 4x^2$ find:

$f(a) = 3 - 5a + 4a^2$

$f(a+h) = 3 - 5(a+h) + 4(a+h)^2 =$

$= 3 - 5a - 5h + 4a^2 + 8ah + 4h^2$

$$\frac{f(a+h) - f(a)}{h} = \frac{[\cancel{3} \cancel{- 5a} - 5h + \cancel{4a^2} + 8ah + 4h^2] - [\cancel{3} \cancel{- 5a} \cancel{+ 4a^2}]}{h}$$

$$= \frac{-5h + 8ah + 4h^2}{h} = \frac{\cancel{h}\,\boxed{(-5 + 8a + 4h)}}{\cancel{h}}$$

The Domain of a Function

The domain of a function is the set of all inputs for the function.

The domain may be stated explicitly.

For example, if we write

$$f(x) = 1 - x^2 \qquad -2 \leq x \leq 5$$

then the domain is the set of all real numbers x for which $-2 \leq x \leq 5$.

If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain is the set of all real numbers for which the expression is defined.*

Fact: Two functions f and g are equal if and only if

1. f and g are defined on the same domain,
2. $f(x) = g(x)$ for all x in the domain.

Example 3:

Find the domain of each of the following functions:

- $f(x) = x^2 + 1$

all x in $\mathbb{R}$

- $g(t) = \dfrac{t^4}{t^2 + t - 6}$ $= \dfrac{t^4}{(t+3)(t-2)}$

domain all $t \in \mathbb{R}$ such that $t \neq -3$ and $t \neq 2$

Example 3 (cont'd):

- $h(u) = \sqrt{7 - 3u}$

all u such that $7 - 3u \geq 0$ $\iff$ $7 \geq 3u$ or $\boxed{u \leq 7/3}$

- $k(x) = \sqrt{\dfrac{x-2}{4-x}}$

all x such that $\dfrac{x-2}{4-x} \geq 0$ $\iff$

$x-2$: $- - - -$ (2) $+ + + + + +$

$4-x$: $+ + + + +$ (4) $- - - -$

$\dfrac{x-2}{4-x}$: $- - -$ (2) $+ + +$ (4) $- - - -$

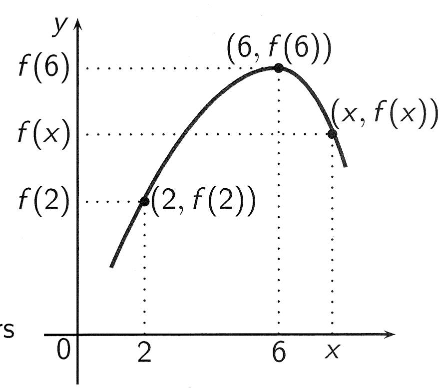
$\therefore$ domain $\boxed{2 \leq x < 4}$

Graphs of Functions

The graph of a function is the most important way to visualize a function. It gives a picture of the behavior or 'life history' of the function. We can read the value of $f(x)$ from the graph as being the height of the graph above the point x.

If f is a function with domain A, then the graph of f is the set of ordered pairs

graph of $f = \{(x, f(x)) \mid x \in A\}$.

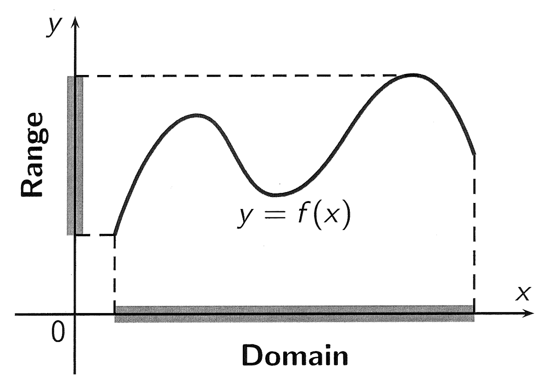


In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

Obtaining Information from the Graph of a Function

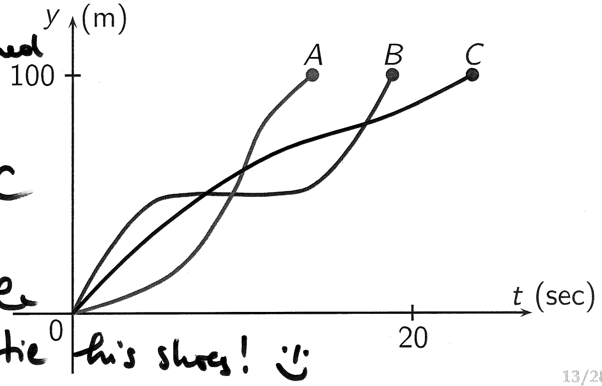
The values of a function are represented by the height of its graph above the x-axis. So, we can read off the values of a function from its graph.

In addition, the graph of a function helps us picture the domain and range of the function on the x-axis and y-axis as shown in the picture:



Example 4 (Hurdle Race):

Three runners compete in a 100-meter hurdle race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race? What do you think happened to runner B?



All three runners finished the race.

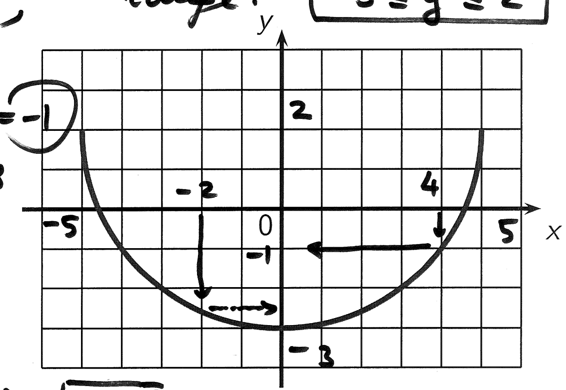
A won the race; B finished second and C came in third

B stopped for a while ... maybe he had to tie his shoes! :)

Example 5:

The picture shows the graph of $g(x) = 2 - \sqrt{25 - x^2}$.
From the graph, find the domain and range of g.

domain: $\boxed{-5 \le x \le 5}$; range: $\boxed{-3 \le y \le 2}$



Find $g(4)$ and $g(-2)$.

$g(4) = 2 - \sqrt{25-4^2} = 2 - 3 = -1$

$g(-2) = 2 - \sqrt{25-2^2} \cong -2.58$

If $g(x) = -2$, what is x?

We can look at the picture or solve

$-2 = 2 - \sqrt{25-x^2} \iff \sqrt{25-x^2} = 4 \iff \boxed{x = \pm 3}$

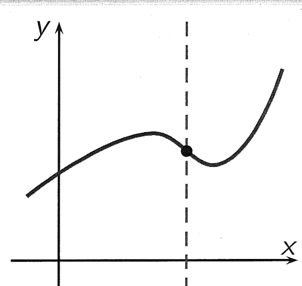
$25 - x^2 = 16 \iff x^2 = 9 \iff$

The Vertical Line Test

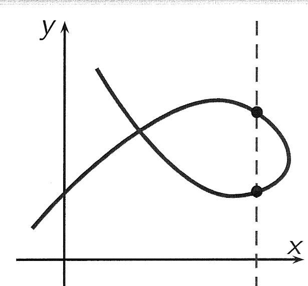
The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions?

The Vertical Line Test

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.



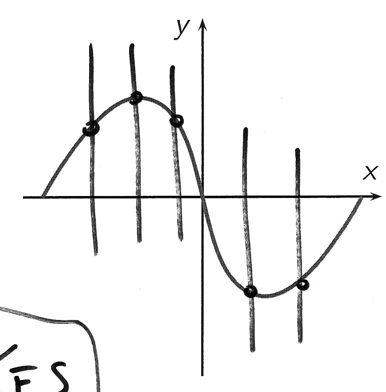
Graph of a function



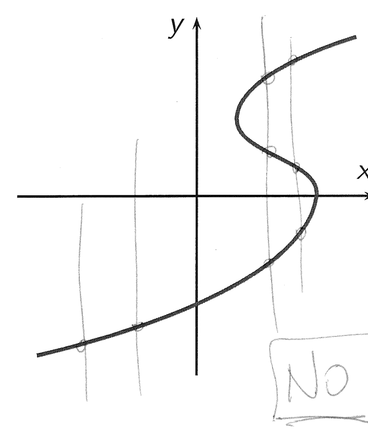
Not a graph of a function

Example 6:

Determine which of the curves drawn below is the graph of a function of x.



YES



NO

Equations that Define Functions

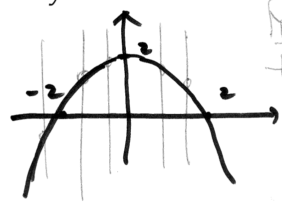
Not every equation in two variables (say x and y) defines one of the variables as a function of the other (say y as a function of x).

Example 9:

Which of the equations that follow define y as a function of x?

$x^2 + 2y = 4$

$\iff y = -\frac{1}{2}x^2 + 2$



YES

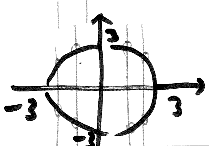
the graph is a parabola

$x = y^2$

No

$x^2 + y^2 = 9$

No



it is a circle

Basic Functions

We introduce the basic functions that we will consider throughout the remainder of the week/semester.

- **polynomial functions**
  A polynomial function is a function of the form
  $$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$
  where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are (real) constants with $a_n \neq 0$. The coefficient a_n is called the leading coefficient, and n is called the degree of the polynomial function. The largest possible domain of f is $\mathbb{R}$.

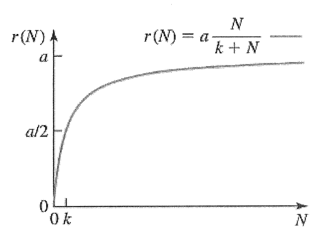
  Examples Suppose a, b, c, and m are constants.

  - Constant functions: $f(x) = c$ (graph is a horizontal line);
  - Linear functions: $f(x) = mx + b$ (graph is a straight line);
  - Quadratic functions: $f(x) = ax^2 + bx + c$ (graph is a parabola).

- **rational functions**
  A rational function is the quotient of two polynomial functions $p(x)$ and $q(x)$: $\quad f(x) = \dfrac{p(x)}{q(x)}$ for $q(x) \neq 0$.

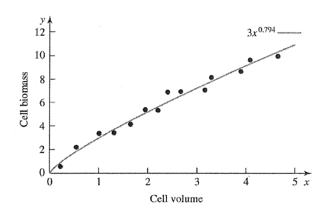
  Example The **Monod growth function** is frequently used to describe the per capita growth rate of organisms when the rate depends on the concentration of some nutrient and becomes saturated for large enough nutrient concentrations.

  If we denote the concentration of the nutrient by N, then the per capita growth rate $r(N)$ is given by
  $$r(N) = \frac{aN}{k+N}, \qquad N \geq 0$$
  where a and k are positive constants.



- **power functions**
  A power function is of the form $f(x) = x^r$ where r is a real number.

  Example Power functions are frequently found in "scaling relations" between biological variables (e.g., organ sizes).
  Finding such relationships is the objective of **allometry**. For example, in a study of 45 species of unicellular algae, a relationship between cell volume and cell biomass was sought. It was found [see, Niklas (1994)] that
  $$\text{cell biomass} \propto (\text{cell volume})^{0.794}$$
  Most scaling relations are to be interpreted in a statistical sense; they are obtained by fitting a curve to data points. The data points are typically scattered about the fitted curve given by the scaling relation.



- **exponential and logarithmic functions**
- **trigonometric functions**

Lines/Linear Functions

If a line lies in a coordinate plane, then the **run** is the change (say Δx) in the x-coordinate and the **rise** is the corresponding change (say Δy) in the y-coordinate between any two points on the line.

The Slope of a Line

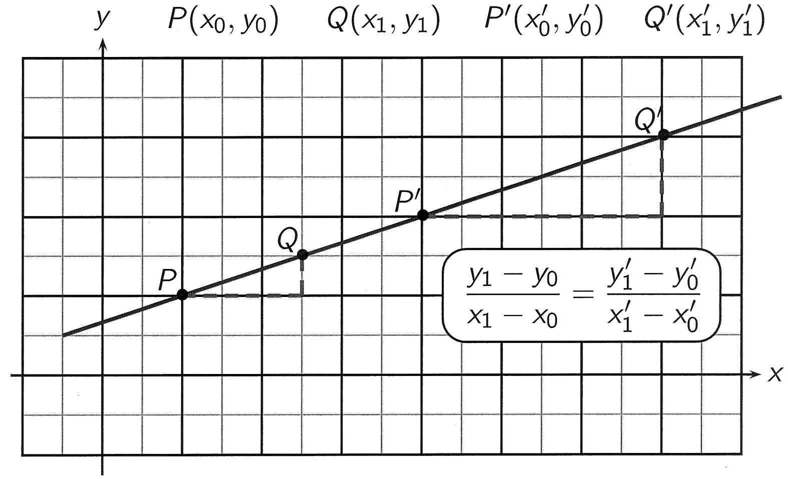
By definition, the slope m of a non-vertical line that passes through the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

Note: The slope of a vertical line is not defined.

Lines/Linear Functions

Note: Properties of similar triangles show that the slope is independent of which two points are chosen on the line.

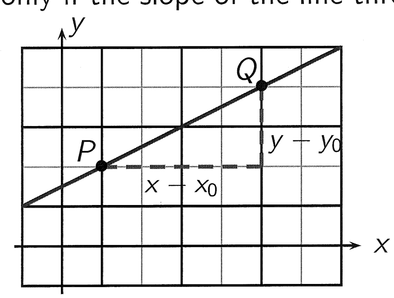


$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y'_1 - y'_0}{x'_1 - x'_0}$$

Equations of Lines

Our goal is to find the equation of a line that passes through a given point $P(x_0, y_0)$ and has slope m. Any point $Q(x, y)$ with $x \neq x_0$ lies on this line if and only if the slope of the line through P and Q is equal to m:

$$\frac{y - y_0}{x - x_0} = m$$



Thus we have:

Point-Slope Form of the Equation of a Line

The line through the point $P(x_0, y_0)$ and with slope m has equation

$$\boxed{y - y_0 = m(x - x_0)} \quad \text{or} \quad \boxed{y = y_0 + m(x - x_0)}$$

Slope-Intercept Form of the Equation of a Line

The line that has slope m and y-intercept b has equation

$$\boxed{y = mx + b}$$

General Equation of a Line

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

Vertical Lines

An equation of the vertical line through (a, b) is $x = a$

Horizontal Lines

An equation of the horizontal line through (a, b) is $y = b$

Parallel and Perpendicular Lines

Parallel Lines

Two lines are parallel if and only if they have the same slope.

Perpendicular Lines

Two lines with slopes m_1 and m_2 are perpendicular if and only if

$$m_1 m_2 = -1 \quad \Leftrightarrow \quad m_2 = -\frac{1}{m_1}.$$

Note: Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

Functions
Basic Functions
Lines/Linear Functions
Various Forms of the Equation of a Line
Parallel and Perpendicular Lines
Examples

Example 7:

- Find an equation of the line that has y-intercept 6 and is parallel to the line $2x + 3y + 4 = 0$. has slope $-2/3$

$\therefore$ $\boxed{y = -\frac{2}{3}x + 6}$

- Find an equation of the line through $(-1, 2)$ and perpendicular to the line $4x - 8y = 1$. slope is $1/2$

$\therefore$ perpendicular line has slope $\boxed{-2}$

$y - 2 = -2(x - (-1)) \iff \boxed{y - 2 = -2(x+1)}$

OR $y = -2x$

Functions
Basic Functions
Lines/Linear Functions
Various Forms of the Equation of a Line
Parallel and Perpendicular Lines
Examples

Example 8 (Global Warming):

Some scientists believe that the average surface temperature of the world has been rising steadily. Suppose that the average surface temperature is given by

$$T = 0.02t + 8.50,$$

where T is the temperature in $^\circ C$ and t is years since 1900.

(a) What do the slope and T-intercept represent?

8.5 is the temperature at $t=0$, i.e. in 1900
The slope 0.02 represents the change in average surface temperature in $^\circ C$ per year

(b) Use the equation to predict the average global surface temperature in 2100.

Since $t=0 \longleftrightarrow 1900$ then $t = 200 \longleftrightarrow 2100$

Thus $T = 0.02(200) + 8.5 = \boxed{12.5\,^\circ C}$

Example 9 (Problem #52, Section 1.2, p. 16):

The Celsius scale is devised so that 0°C is the freezing point of water (at 1 atmosphere of pressure) and 100°C is the boiling point of water (at 1 atmosphere of pressure).
If you are more familiar with the Fahrenheit scale, then you know that water freezes at 32°F and boils at 212°F.

(a) Find a linear equation/function that relates temperature measured in degrees Celsius and temperature measured in degrees Fahrenheit.

(b) The normal body temperature in humans ranges from 97.6°F to 99.6°F. Convert this temperature range into degrees Celsius.

$F = aC + b$

* when $C = 0$ then $F = 32$ so that

$32 = a \cdot 0 + b \implies \boxed{b = 32}$

* when $C = 100$ then $F = 212$ so that

$212 = a \cdot 100 + 32 \implies$

$100a = 212 - 32 \qquad a = \frac{180}{100} = \frac{9}{5}$

$\therefore \boxed{F = \frac{9}{5}C + 32}$

Alternatively we can solve for C

$F = \frac{9}{5}C + 32 \iff 5F = 9C + 160$

$C = \frac{5}{9}F - \frac{160}{9} \cong \frac{5}{9}F - 17.7$

(b) $\boxed{97.6 \le F \le 99.6}$

range of normal body temperature in humans

substitute

$97.6 \le \frac{9}{5}C + 32 \le 99.6$

$\iff 97.6 - 32 \le \frac{9}{5}C \le 99.6 - 32$

$\iff 65.6 \le \frac{9}{5}C \le 67.6$

$\iff \frac{5}{9} \cdot 65.6 \le C \le \frac{5}{9} \cdot 67.6$

$\iff 36.44 \le C \le 37.55$