FastTrack — MA 137/MA 113 — BioCalculus

Functions (2): More Examples and Transformations of Functions

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Goal: `We continue with more examples of basic functions. We also study how certain transformations (≡shifting, reflecting, and stretching) of a function affect its graph. This gives us a better understanding of how to graph functions.`

Parabolas/Quadratic Functions

A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c,$$

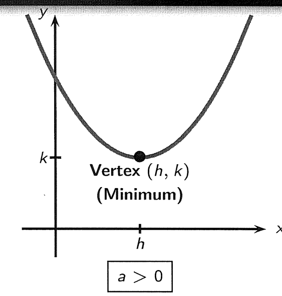
where a, b, and c are real numbers and $a \neq 0$.

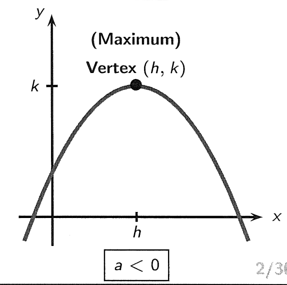
The graph of any quadratic function is a parabola; it can be obtained from the graph of $f(x) = x^2$ by elementary transformations.

Indeed, by completing the square, a quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k.$$

The graph of f is a parabola with vertex (h, k); the parabola opens upward if $a > 0$, or downward if $a < 0$.





Expressing a quadratic function in standard form helps us sketch its graph and find its maximum or minimum value.
There is a **formula** for (h, k) that can be derived from the general quadratic function as follows:

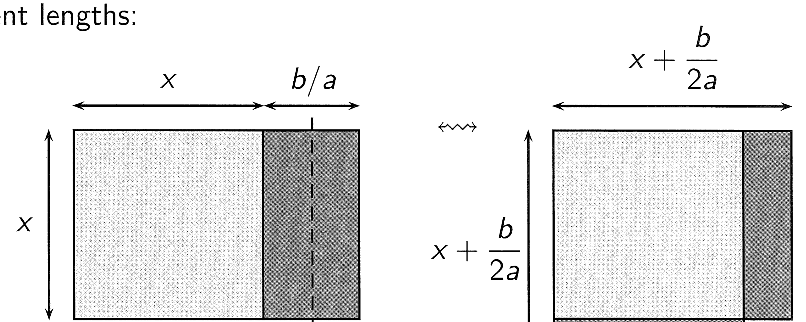
$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

Thus:

$$\boxed{h = -\frac{b}{2a} \qquad k = \frac{4ac - b^2}{4a}}$$

Geometric Interpretation of Completing the Square

This interpretation goes back to the Babylonian scribes, who fully used the "cut-and-paste" geometry developed by the ancient surveyors (ca. 1700 BC). Here, x, a, and b are positive as they represent lengths:



where ▢ is a square of side $b/2a$; thus its area is $(b/2a)^2$.

The Quadratic Formula

The previous calculation actually allows us to derive the general formula for the solution of the quadratic equation:

The Quadratic Formula

The roots x_1 and x_2 of the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0$, are:

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Note: The easiest method to solve a quadratic equation is by factoring it.

Use the quadratic formula only when a factorization is not readily visible.

Example 1 (Torricelli's Law):

A tank holds 50 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 20 minutes. The tank drains faster when it is nearly full because the pressure on the leak is greater. **Torricelli's Law** gives the volume of the water remaining in the tank after t minutes as

$$V(t) = 50\left(1 - \frac{t}{20}\right)^2 \qquad 0 \leq t \leq 20.$$

(a) Find $V(0)$ and $V(20)$.
(b) What do your answers to part **(a)** represent?
(c) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.

(a) $V(0) = 50\left(1 - \frac{0}{20}\right)^2 = 50 \cdot 1^2 = \underline{\underline{50}}$

$V(20) = 50\left(1 - \frac{20}{20}\right)^2 = 50 \cdot 0^2 = \underline{\underline{0}}$

(b) at $t=0$ the tank is full;
at $t=20$ the tank is empty

(c)

t	0	5	10	15	20
$V(t)$	50	28.125	12.5	3.125	0

Example 2:

When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by

$$C(t) = 0.06t - 0.0002t^2,$$

where $0 \leq t \leq 240$ and the concentration is measured in mg/L.
When is the maximum serum concentration reached?
What is that maximum concentration?

Consider $C(t) = 0.06t - 0.0002t^2$

and rewrite it as

$$C(t) = -0.0002t^2 + 0.06t$$

We want to complete the squares

$$C(t) = -0.0002\left(t^2 - \frac{0.06}{0.0002}t\right) = -0.0002\left(t^2 - 300t\right)$$

$$= -0.0002\left(t^2 - 300t + \underline{\underline{\left(\tfrac{300}{2}\right)^2}}\right) \ \underline{\underline{+\ 4.5}}$$

notice $4.5 = 0.0002 \cdot \left(\frac{300}{2}\right)^2$

$\therefore$ max serum concentration at $t = 150$

$$\therefore \boxed{C(t) = -0.0002(t-150)^2 + 4.5}$$

$\therefore$ max conc. 4.5

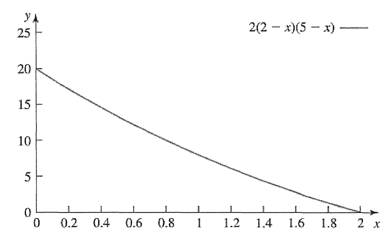
A Chemical Reaction (Example 5, Section 1.3, p. 22)

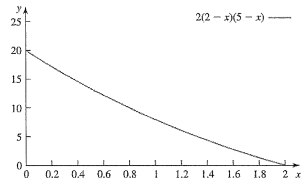
- Consider the reaction rate of the chemical reaction
$$A + B \longrightarrow AB$$
in which the molecular reactants A and B form the molecular product AB.
- The rate at which this reaction proceeds depends on how often A and B molecules collide.
- The **law of mass action** states that the rate at which this reaction proceeds is proportional to the product of the respective concentrations of the reactants. (Here, concentration means the number of molecules per fixed volume.)
- Denote the reaction rate by R and the concentration of A and B by [A] and [B], respectively. The law of mass action says that
$$R \propto [A]\cdot[B]$$
- Introduce the proportionality factor k. We obtain $R = k[A]\cdot[B]$.

- Note that $k > 0$, because [A], [B], and R are positive.
- We assume now that the reaction occurs in a closed vessel; that is, we add specific amounts of A and B to the vessel at the beginning of the reaction and then let the reaction proceed without further additions.
- We can express the concentrations of the reactants A and B during the reaction in terms of their initial concentrations a and b and the concentration of the molecular product [AB].
- If $x = [AB]$, then
$$[A] = a - x \quad \text{for} \quad 0 \le x \le a \quad \text{and} \quad [B] = b - x \quad \text{for} \quad 0 \le x \le b.$$
- The concentration of AB cannot exceed either of the concentrations of A and B.
(For example, suppose five A molecules and seven B molecules are allowed to react; then a maximum of five AB molecules can result, at which point all of the A molecules are used up and the reaction ceases. The two B molecules left over have no A molecules to react with.)

- Therefore, we get
$$R(x) = k(a-x)(b-x) \quad \text{for} \quad 0 \le x \le a \quad \text{and} \quad 0 \le x \le b.$$
- The condition $0 \le x \le a$ and $0 \le x \le b$ can be written as $0 \le x \le \min(a, b)$.
- Expand the expression for $R(x)$, to see that $R(x)$ is indeed a polynomial function (of degree 2)
$$R(x) = k(ab - ax - bx + x^2) = kx^2 - k(a+b)x + kab$$
for $0 \le x \le \min(a, b)$.

A graph of $R(x)$, $0 \le x \le a$, is shown for the case $a \le b$.
(We chose $k = 2$, $a = 2$, and $b = 5$.)



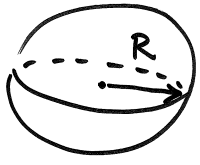


- Notice that when $x = 0$ (i.e., when no AB molecules have yet formed), the rate at which the reaction proceeds is at a maximum.
- As more and more AB molecules form and, consequently, the concentrations of the reactants decline, the reaction rate decreases.
- This should also be intuitively clear: As fewer and fewer A and B molecules are in the vessel, it becomes less and less likely that they will collide to form the molecular product AB.
- When $x = a = \min(a, b)$, the reaction rate $R(a) = 0$. This is the point at which all A molecules are exhausted and the reaction necessarily ceases.

Example 3:

Find the scaling relation between the surface area S and the volume V of a sphere of radius R.

[More precisely, show that $S = (36\pi)^{1/3}\, V^{2/3}$, that is, $S \propto V^{2/3}$.]



Recall that the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$

The surface area of a sphere of radius R is $S = 4\pi R^2$. We want to write S as a function of V.

FROM: $V = \frac{4}{3}\pi R^3 \rightsquigarrow \frac{3}{4\pi}V = R^3$

so that $R = \left(\frac{3}{4\pi}V\right)^{1/3}$. Substitute in $S = 4\pi R^2$ to get $S = 4\pi\left[\left(\frac{3}{4\pi}V\right)^{1/3}\right]^2$

$$\therefore \quad S = \underbrace{4\pi\left(\frac{3}{4\pi}\right)^{2/3}}_{} \cdot V^{2/3}$$

$$\left(64\pi^3\,\frac{9}{16\pi^2}\right)^{1/3} \cdot V^{2/3}$$

$$= (36\pi)^{1/3} \cdot V^{2/3}$$

i.e. $S \propto V^{2/3}$

Example 4: (Michaelis-Menten enzymatic reaction)

According to the Michaelis-Menten equation (1913) when a chemical reaction involving a substrate S is catalyzed by an enzime, the rate of reaction $V = V([S])$ is given by the expression

$$V = \frac{V_{\max}[S]}{K_m + [S]},$$

where [S] denotes substrate concentration (for examples in moles per liter), and $V_{\max}$ and K_m are constants.

$V_{\max}$ is the maximal velocity of the reaction and K_m is the Michaelis constant.
K_m is the substrate concentration at which the reaction achieves half of the maximum velocity.

Graph V assuming that $V_{\max} = 3$ and $K_m = 2$. That is,

$$V = \frac{3[S]}{2 + [S]}.$$

Example 5: (Lineweaver-Burk plot)

The Lineweaver-Burk plot (1934) was widely used to determine important terms in enzyme kinetics, such as K_m and V_{max}, before the wide availability of powerful computers and non-linear regression software.

The Michaelis-Menten rate function $V = \dfrac{V_{\max}[S]}{K_m + [S]}$ traces out a hyperbola. The *reciprocal of this expression* is written

$$\frac{1}{V} = \frac{K_m}{V_{\max}} \frac{1}{[S]} + \frac{1}{V_{\max}}$$

That is, the reciprocal expression is linear in $x = \dfrac{1}{[S]}$ and $y = \dfrac{1}{V}$.

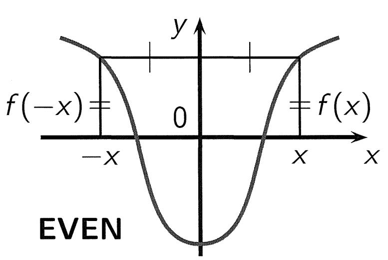
The slope of this line is $K_m/V_{\max}$;
the y-intercept is $1/V_{\max}$ and the x-intercept is $-1/K_m$.

The graph in the xy-plane is called the Lineweaver-Burk plot.

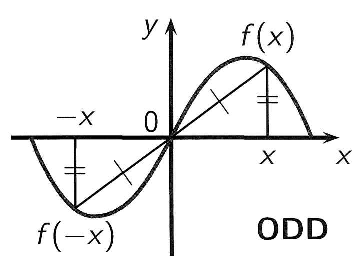
Eg: Given $V = \dfrac{3[S]}{2+[S]}$, plot $y = \dfrac{2}{3}x + \dfrac{1}{3}$.

Even and Odd Functions

Let f be a function.
f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



Graph symmetric wrt y-axis.



Graph symmetric wrt $(0,0)$.

Example 6:

Determine whether the following functions are even or odd:

$f(x) = x^3 + 2x^5$

$$f(-x) = (-x)^3 + 2(-x)^5 = -x^3 - 2x^5 =$$
$$= -(x^3 + 2x^5) = -f(x) \quad \therefore \underline{\underline{\text{ODD}}}$$

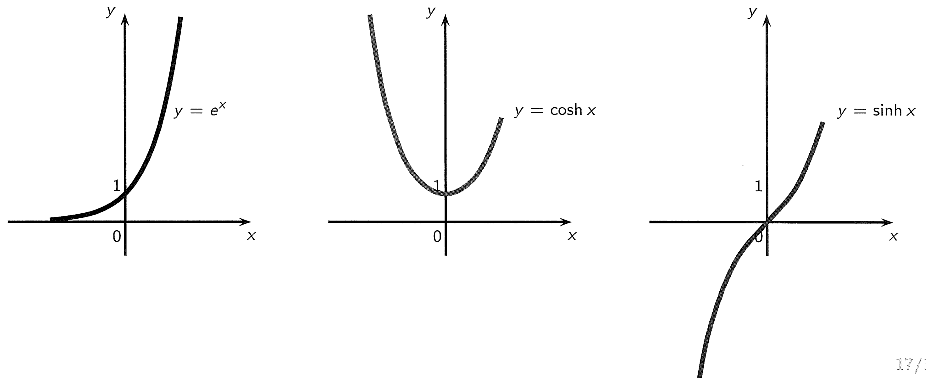
$g(x) = x^2 - 3x^4$

$$g(-x) = (-x)^2 - 3(-x)^4 = x^2 - 3x^4 = g(x)$$
$$\therefore \underline{\underline{\text{EVEN}}}$$

Curious/Amazing Fact!

Any function can be uniquely written as an even plus an odd function.

Example: $$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\cosh x} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\sinh x}$$

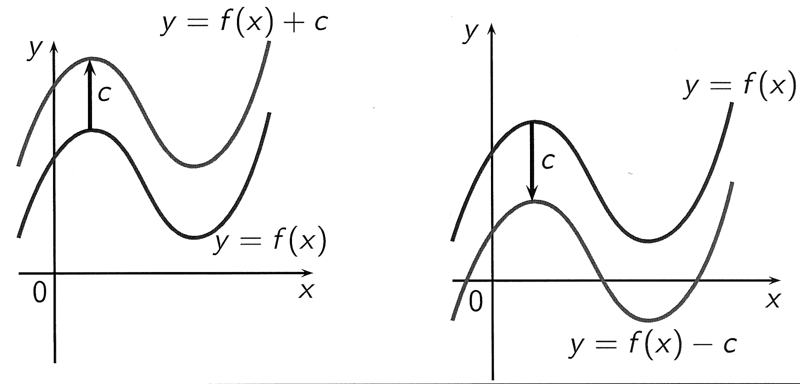


Transformations of Functions

Vertical Shifting: Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.

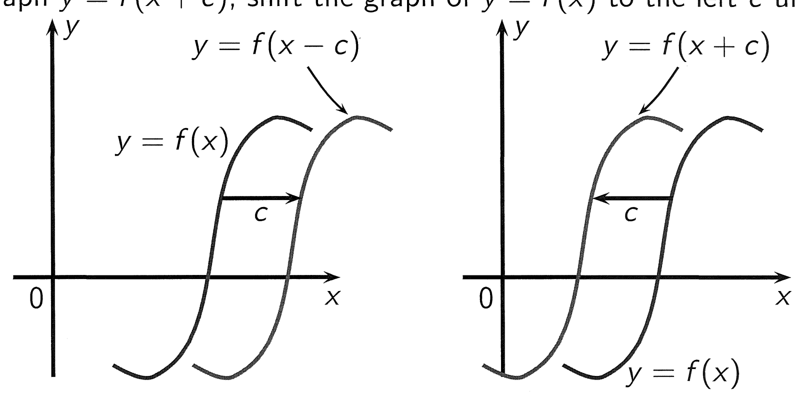


Transformations of Functions

Horizontal Shifting: Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.

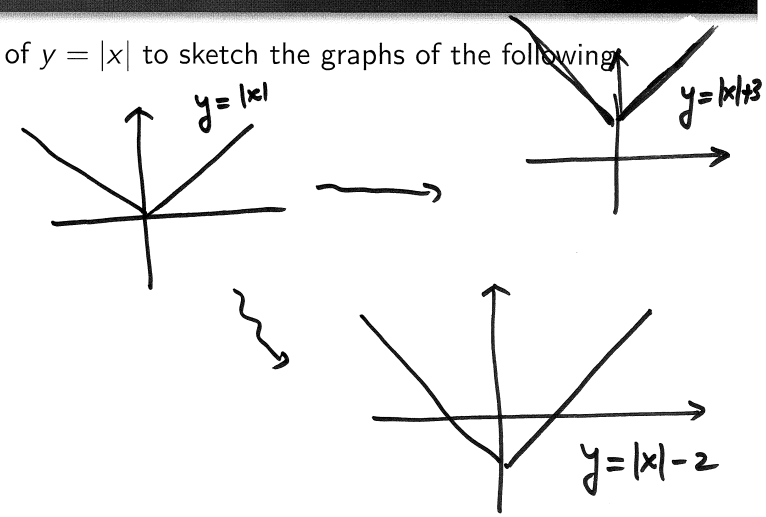


Example 7:

Use the graph of $y = |x|$ to sketch the graphs of the following functions:

$y = |x| + 3$

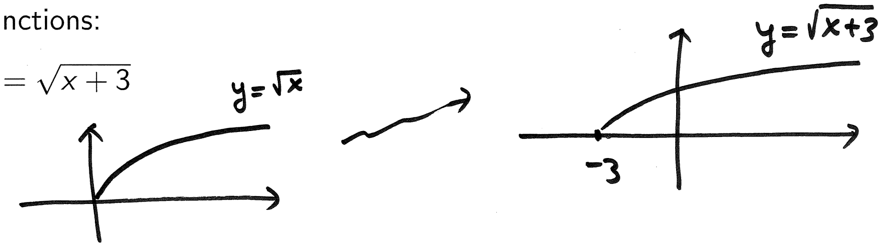
$y = |x| - 2$



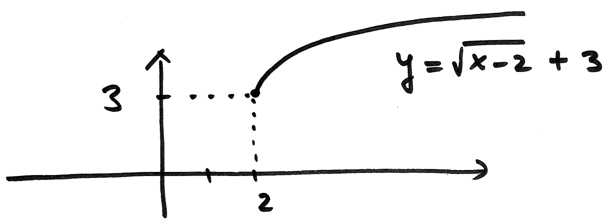
Example 8:

Use the graph of $y = \sqrt{x}$ to sketch the graphs of the following functions:

$y = \sqrt{x+3}$

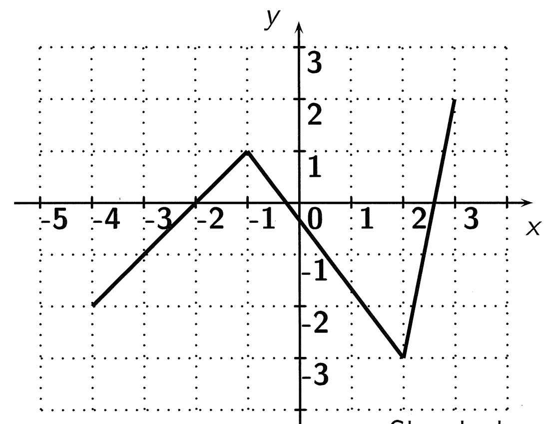


$y = \sqrt{x-2} + 3$



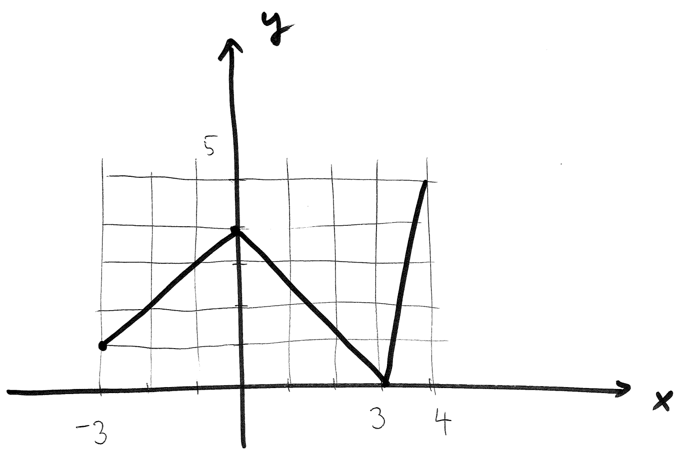
Example 9:

The graph of $y = f(x)$ is shown below.



Sketch the graph of $y = f(x-1) + 3$.

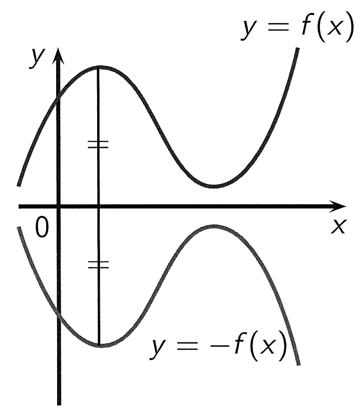
$f(x-1)+3$ move graph of f of one unit to the right and 3 units up

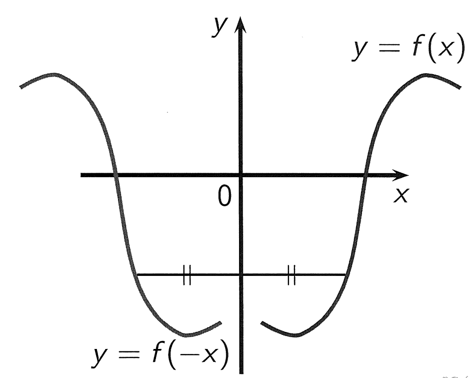


Reflecting Graphs

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

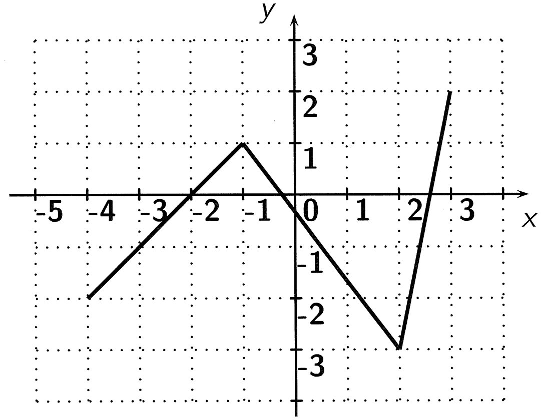
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.





Example 10:

The graph of $y = f(x)$ is shown below.



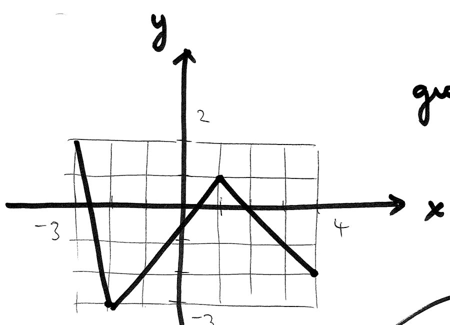
Basic Functions (cont'd)
Transformations of Functions
Vertical Shifting
Horizontal Shifting
Reflecting Graphs
Vertical Stretching and Shrinking
Horizontal Shrinking and Stretching

Example 10 (cont'd):

Sketch the graph of $y = f(-x)$.

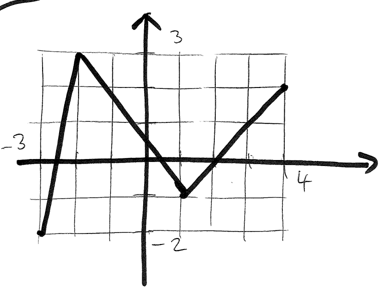
Sketch the graph of $y = -f(x)$.

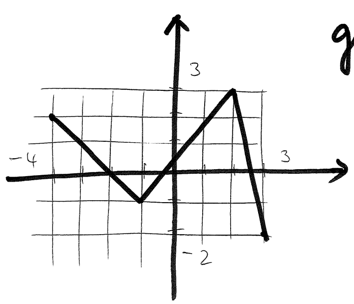
Sketch the graph of $y = -f(-x)$.



graph of f(-x)

graph of -f(x)





graph of -f(-x)

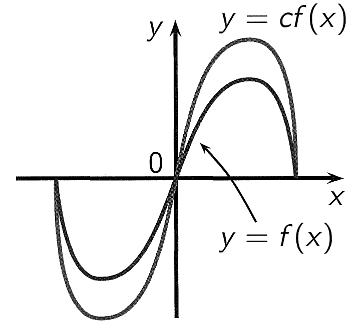
Basic Functions (cont'd)
Transformations of Functions
Vertical Shifting
Horizontal Shifting
Reflecting Graphs
Vertical Stretching and Shrinking
Horizontal Shrinking and Stretching

Transformations of Functions

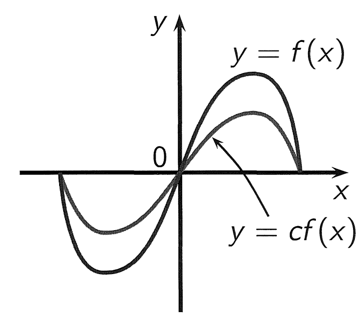
Vertical Stretching and Shrinking: To graph $y = cf(x)$:

If $c > 1$, STRETCH the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, SHRINK the graph of $y = f(x)$ vertically by a factor of c.



$c > 1$



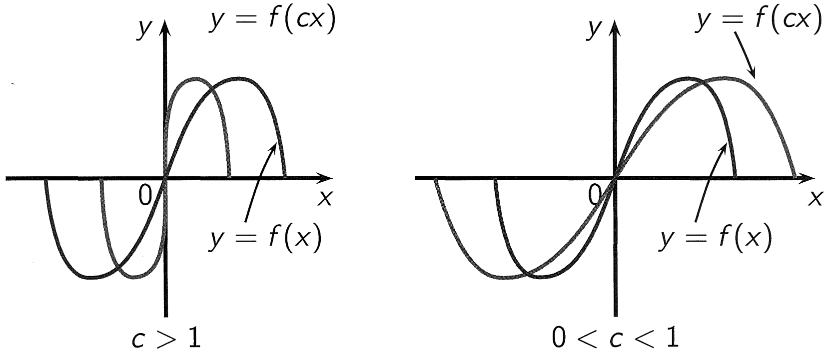
$0 < c < 1$

Transformations of Functions

Horizontal Shrinking and Stretching: To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

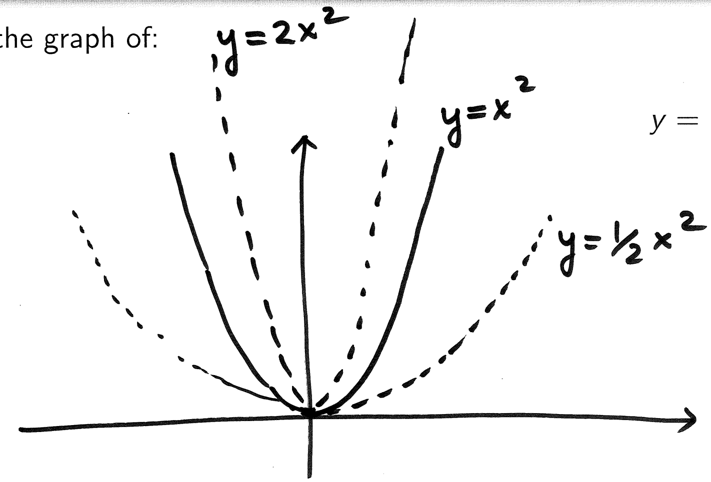
If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.



Example 11:

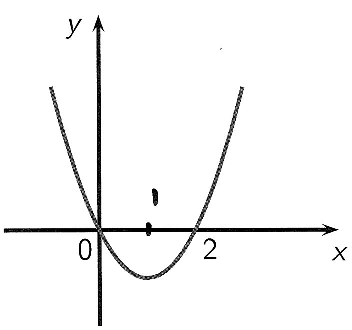
Sketch the graph of:

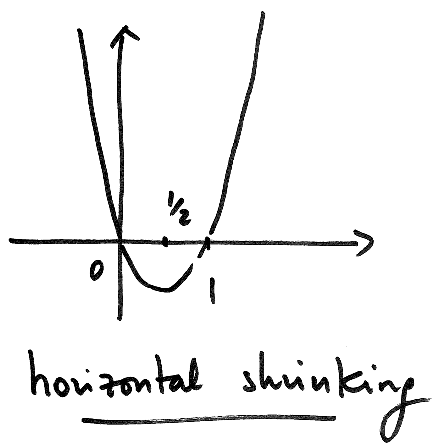
$y = 2x^2$ $\qquad$ $y = \frac{1}{2}x^2$



Example 12:

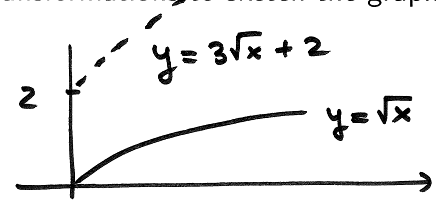
Use the graph of $f(x) = x^2 - 2x$ provided below to sketch the graph of $f(2x)$.





Example 13:

Use transformations to sketch the graph of $y = 3\sqrt{x} + 2$.



y = 3√x + 6

Use transformations to sketch the graph of $y = 3(\sqrt{x} + 2)$.

y = 3√x + 6