FastTrack — MA 137/MA 113 — BioCalculus

Functions (4): Exponential and Logarithmic Functions

Alberto Corso – ⟨alberto.corso@uky.edu⟩

Department of Mathematics – University of Kentucky

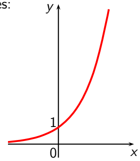
Goal: We introduce two new classes of functions called *exponential and logarithmic functions*. They are inverses of each other. Exponential functions are appropriate for modeling such natural processes as population growth for all living things and radioactive decay.

Exponential Functions

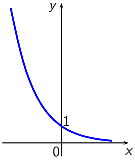
The **exponential function**

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

Example 1:

Let $f(x) = 2^x$. Evaluate the following:

$f(2) =$

$f(-1/3) =$

$f(\pi) =$

$f(-\sqrt{3}) =$

Example 2:

Draw the graph of each function:

$f(x) = 2^x$

$g(x) = \left(\frac{1}{2}\right)^x$

Example 3:

Use the graph of $f(x) = 3^x$ to sketch the graph of each function:

$g(x) = 1 + 3^x$

$h(x) = -3^x$

$k(x) = 2 - 3^{-x}$

The Number 'e'

The most important base is the number denoted by the letter e.
The number e is defined as the value that $(1+1/n)^n$ approaches as n becomes very large.
Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

The Natural Exponential Function

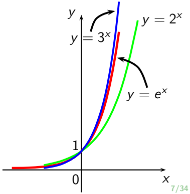
The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



Example 4:

When a certain drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50\, e^{-0.2t}.$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

Compound Interest

Compound interest is calculated by the formula:

$$P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

where

$P(t)$ = principal after t years
P_0 = initial principal
r = interest rate per year
n = number of times interest is compounded per year
t = number of years

Continuously Compounded Interest

Continuously compounded interest is calculated by the formula:

$$P(t) = P_0\, e^{rt}$$

where

$P(t)$ = principal after t years P_0 = initial principal
r = interest rate per year t = number of years

Proof: The interest paid increases as the number n of compounding periods increases. If $m = \frac{n}{r}$, then:

$$P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

But as m becomes large, the quantity $(1 + 1/m)^m$ approaches the number e. Thus, we obtain the formula for the continuously compounded interest.

Example 5:

Suppose you invest $\$2,000$ at an annual rate of 12% ($r = 0.12$) compounded quarterly ($n = 4$). How much money would you have one year later? What if the investment was compounded monthly ($n = 12$)?

Example 6:

Suppose you invest $\$2,000$ at an annual rate of 9% ($r = 0.09$) compounded continuously. How much money would you have after three years?

Logarithmic Functions

Every exponential function $f(x) = a^x$, with $0 < a \neq 1$, is a one-to-one function (Horizontal Line Test). Thus, it has an inverse function, called the *logarithmic function with base a* and denoted by $\log_a x$.

Definition

Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a x \iff a^y = x.$$

In other words, $\log_a x$ is the exponent to which a must be raised to give x.

Properties of Logarithms

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

Example 7:

Change each exponential expression into an equivalent expression in logarithmic form:

$5^3 = b$

$a^6 = 15$

$e^{t+1} = 0.5$

Example 8:

Change each logarithmic expression into an equivalent expression in exponential form:

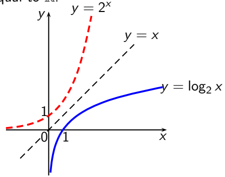
$\log_3 81 = 4$

$\log_8 4 = \frac{2}{3}$

$\log_e(x - 3) = 2$

Graphs of Logarithmic Functions

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Thus, the function $y = \log_a x$ is defined for $x > 0$ and has range equal to $\mathbb{R}$.



The point $(1, 0)$ is on the graph of $y = \log_a x$ (as $\log_a 1 = 0$) and the y-axis is a vertical asymptote.

Example 9:

Find the domain of the function $f(x) = \log_3(x+2)$ and sketch its graph.

Common Logarithms

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base: $\boxed{\log x := \log_{10} x.}$

Example 10 (Bacteria Colony):

A certain strain of bacteria divides every three hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\,\frac{\log(N/50)}{\log 2}.$$

Find the time required for the colony to grow to a million bacteria.

Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition

The logarithm with base e is called the **natural logarithm** and denoted:

$$\boxed{\ln x := \log_e x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \qquad \Longleftrightarrow \qquad e^y = x.$$

Properties of Natural Logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Example 11:

Evaluate each of the following expressions:

$\ln e^9$

$\ln \dfrac{1}{e^4}$

$e^{\ln 2}$

Example 12:

Graph the function $y = 2 + \ln(x - 3)$.

Example 13:

Find the domain of the function $f(x) = 2 + \ln(10 + 3x - x^2)$.

Laws of Logarithms

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

1. $\log_a(AB) = \log_a A + \log_a B$;

2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$;

3. $\log_a(A^C) = C \log_a A$.

Proof of **Law 1.**: $\log_a(AB) = \log_a A + \log_a B$

Let us set

$$\log_a A = u \quad \text{and} \quad \log_a B = v.$$

When written in exponential form, they become

$$a^u = A \quad \text{and} \quad a^v = B.$$

Thus:

$$\begin{aligned} \underline{\log_a(AB)} &= \log_a(a^u\, a^v) \\ &= \log_a(a^{u+v}) \\ &\overset{\text{why?}}{=} u + v \\ &= \underline{\log_a A + \log_a B}. \end{aligned}$$

In a similar fashion, one can prove **2.** and **3.**

Example 14:

Evaluate each expression:

$\log_5 5^9$ $\qquad \log_3 7 + \log_3 2$ $\qquad \log_3 16 - 2\log_3 2$

$\ln\left(\ln e^{(e^{200})}\right)$ $\qquad \log_3 100 - \log_3 18 - \log_3 50$

Expanding and Combining Logarithmic Expressions

Example 15:

Use the Laws of Logarithms to expand each expression:

$\log_2(2x)$

$\log_5(x^2(4-5x))$

$\log\left(x\sqrt{\frac{y}{z}}\right)$

Example 16:

Use the Laws of Logarithms to combine the expression

$$\log_a b + c\log_a d - r\log_a s$$

into a single logarithm.

Example 17:

Use the Laws of Logarithms to combine the expression

$$\ln 5 + \ln(x+1) + \frac{1}{2}\ln(2-5x) - 3\ln(x-4) - \ln x$$

into a single logarithm.

Example 18 (Forgetting):

Ebbinghaus's Law of Forgetting states that if a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c \log(t+1),$$

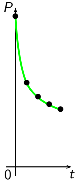
where c is a constant that depends on the type of task and t is measured in months.

(a) Solve the equation for P.

(b) Use Ebbinghaus's Law of Forgetting to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume $c = 0.3$.

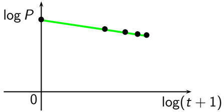
Comment (about Example 18)

t	$P = 80/(t+1)^{0.3}$
0	80
6	44.62
12	37.06
18	33.072
24	30.458



Comment (cont.d)

t	$\log(t+1)$	$\log P = \log 80 - 0.3 \log(t+1)$
0	0	1.903
6	0.845	1.650
12	1.114	1.569
18	1.279	1.519
24	1.398	1.484



Example 19 (Biodiversity):

Some biologists model the number of species S in a fixed area A (such as an island) by the **Species-Area relationship**

$$\log S = \log c + k \log A,$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part **(a)** to show that if $k = 3$ then doubling the area increases the number of species eightfold.

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}.}$$

Proof: Set $y = \log_b x$. By definition, this means that $b^y = x$. Apply now $\log_a(\cdot)$ to $b^y = x$. We obtain

$$\log_a(b^y) = \log_a x \qquad \rightsquigarrow \qquad y \log_a b = \log_a x.$$

Thus

$$\log_b x = y = \frac{\log_a x}{\log_a b}.$$

Exponential Functions
Logarithmic Functions
Definition
Graphs of Logarithmic Functions
Common and Natural Logarithms
Laws of Logarithms
Base Change

Example 20:

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct up to five decimal places:

$\log_5 2$

$\log_4 125$

$\log_{\sqrt{3}} 5$