MA 137: Calculus I for the Life Sciences

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Section 5.6: Difference Equations: Cobwebbing

- We can determine graphically whether a fixed point is stable or unstable.
- The fixed points of exponential growth recursive sequence are found graphically where the graphs of $N_{t+1} = RN_t$ and $N_{t+1} = N_t$ intersect.
- We see that the two graphs intersect where $N_t = 0$ only when $R \neq 1$.

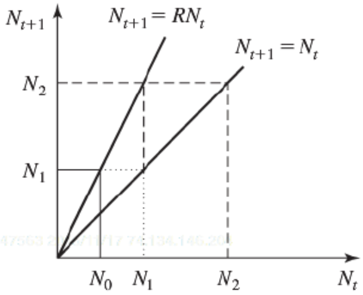

Figure: Cobwebbing for the exponential model.

We can use the two graphs on the left to follow successive population sizes. Start at N_0 on the horizontal axis. Since $N_1 = RN_0$, we find N_1 on the vertical axis, as shown by the solid vertical and horizontal line segments. Using the line $N_{t+1} = N_t$, we can locate N_1 on the horizontal axis by the dotted horizontal and vertical line segments. Using the line $N_{t+1} = RN_t$ again, we can find N_2 on the vertical axis, as shown in the figure by the broken horizontal and vertical line segments. Using the line $N_{t+1} = N_t$ once more, we can locate N_2 on the horizontal axis and then repeat the preceding steps to find N_3 on the vertical axis, and so on. This procedure is called **cobwebbing**.

Section 5.6: Difference Equations: Cobwebbing

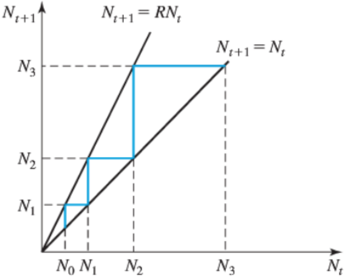

Figure: Cobwebbing for $R > 1$.

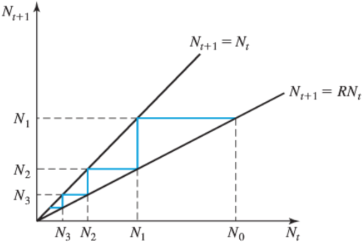

Figure: Cobwebbing for $0 < R < 1$.

If $R > 1$, and we see that if $N_0 > 0$, then N_t will not converge to the fixed point $N^* = 0$, but instead will move away from 0 and, in fact, will go to infinity as t tends to ∞.

If $0 < R < 1$, we see that if $N_0 > 0$, then N_t will return to the fixed point $N* = 0$.

Section 5.6: Cobwebbing: General Case

The general form of a first-order recursion is

$$x_{t+1} = f(x_t), \quad t = 0, 1, 2, \ldots$$

We assume that the function f is differentiable in its domain.

- To find fixed points algebraically, we solve $x = f(x)$.
- To find them graphically, we look for points of intersection of the graphs of $x_{t+1} = f(x_t)$ and $x_{t+1} = x_t$.

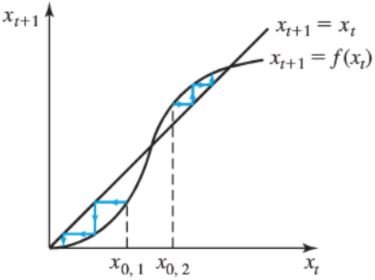

Figure: Cobwebbing for the exponential model.

The graphs in the picture intersect more than once, which means that there are multiple equilibria. We can use the cobwebbing procedure from the previous subsection to graphically investigate the behavior of the difference equation for different initial values.

Two cases are shown in the picture, one starting at $x_{0,1}$ and the other at $x_{0,2}$. We see that x_t converges to different values, depending on the initial value.

Example

(a) Find all the fixed (equilibrium) points for the recursive sequence $x_{t+1} = x_t^2$.

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(*b*) What does the Stability Criterion say about the fixed (equilibrium) points found in part (*a*)?

(*c*) Sketch a cobweb graph starting at $x_0 = 1.1$ and $x_0 = 0.75$, respectively. Use it to determine $\lim_{t \longrightarrow \infty} x_t$ in each case.

In the three examples that follow

- The Beverton-Holt Recruitment Model,
- The Discrete Logistic Equation,
- Ricker Logistic Equation.

we will see that discrete-time population models show very rich and complex behavior.

Earlier, we discussed the exponential growth model defined by the recursion

$$N_{t+1} = RN_t \text{ with } N_0 = \text{ population size at time 0.}$$

When $R > 1$, the population size will grow indefinitely, if $N_0 > 0$. Such growth, called **density-independent growth**, is biologically unrealistic.

$$N_{t+1} = \frac{RN_t}{1 + \frac{R-1}{K}N_t}$$

This recursion is known as the **Beverton-Holt recruitment curve.**

We have two fixed points when $R > 1$:

- the fixed point $\hat{N} = 0$, which we call trivial, since it corresponds to the absence of the population, and
- the fixed point $\hat{N} = K$, which we call nontrivial, since it corresponds to a positive population size.

One can show that, when $K > 0$, $R > 1$, and $N_0 > 0$, we have that

$$\lim_{t\to\infty} N_t = K$$

The most popular discrete-time single-species model is the discrete logistic equation, whose recursion is given by

$$N_{t+1} = N_t \left[1 + R\left(1 - \frac{N_t}{K}\right)\right]$$

where R and K are positive constants. R is called the **growth parameter** and K is called the **carrying capacity.**

This model of population growth exhibits very complicated dynamics, described in an influential review paper by Robert May (1976). We first rewrite the model in what is called the canonical form

$$x_{t+1} = rx_t(1 - x_t)$$

where $r = 1 + R$ and

$$x_t = \frac{R}{K(1+R)} N_t$$

We first compute the fixed points of the discrete logistic equation written in standard form. We need to solve

$$x = rx(1 - x)$$

Solving immediately yields the solution $\hat{x} = 0$. If $x \neq 0$, we divide both sides by x and find that

$$1 = r(1 - x) \text{ or } \hat{x} = 1 - \frac{1}{r}$$

Provided that $r > 1$, both fixed points are in $[0, 1)$.

- The fixed point $\hat{x} = 0$ corresponds to the fixed point $\hat{N} = 0$, which is why we call $\hat{x} = 0$ a trivial equilibrium.
- When $\hat{x} = 1 - 1/r$ we obtain that $\hat{N} = K$ is the other fixed point.

Example (HW 21, Problem # 6)

We investigate the canonical discrete-time logistic growth model

$$x_{t+1} = rx_t(1 - x_t)$$

for $t = 0, 1, 2, \dots$

Show that for $r > 1$, there are two fixed points. For which values of r is the nonzero fixed point locally stable?

Section 5.6: The Ricker Logistic Equation

An iterated map that has the same (desirable) properties as the logistic map but does not admit negative population sizes (provided that N_0 is positive) is **Ricker's curve.** The recursion, called the **Ricker logistic equation**, is given by

$$N_{t+1} = N_t e^{R\left(1-\frac{N_t}{K}\right)}$$

where R and K are positive parameters.

As in the discrete logistic model, R is the growth parameter and K is the carrying capacity. The fixed points are $\hat{N} = 0$ and $\hat{N} = K$.

The Ricker logistic equation shows the same complex dynamics as the discrete logistic map [convergence to the fixed point for small positive values of R, periodic behavior with the period doubling as R increases, and chaotic behavior for larger values of R.

Example (HW 21, Problem # 8)

We consider density-dependent population growth models of the form

$$N_{t+1} = R(N_t)\, N_t$$

The function $R(N) = e^{r(1-N/K)}$ describes the per capita growth.

Find all nontrivial fixed points $\widehat{N}$ (i.e., $\widehat{N} > 0$) and determine the stability as a function of the parameter values. We assume that the function parameters are $r > 0$ and $K > 0$.