FIBONACCI'S NUMBERS, A POPULATION MODEL, AND POWERS OF MATRICES

The goal of these notes is to illustrate an application of large powers of matrices. Our primary tools are the eigenvalues and eigenvectors of the matrix. We illustrate this with two familiar examples.

Fibonacci's numbers. We are all familiar with Fibonacci's sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

What if we wanted to compute 'quickly' (this is the keyword!) the 1000^{th} Fibonacci's number? Here is how matrices can help us.

Write $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, \ldots$ In other words, Fibonacci's numbers are given by the recursive relation $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$, with $f_0 = 0$ and $f_1 = 1$. Notice that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \cdots$$

That is, we can write the previous expressions as

$$\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \qquad \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} \qquad \begin{bmatrix} f_4 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} \qquad \cdots$$

From this it also follows that

$$\begin{bmatrix} f_3 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \qquad \begin{bmatrix} f_4 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \qquad \cdots$$

In general we see that if we set $\mathbf{u}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ we have the *recursive relation*

$$\mathbf{u}_{n+1} = \begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = A\mathbf{u}_n \qquad \mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1}$$

We can also solve (1) explicitly and produce the solution in terms of the powers of the matrix A. That is

$$\mathbf{u}_n = A^n \mathbf{u}_0 \qquad \mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2}$$

Notice that (1) gives us the '*transition*' (relation) between consecutive Fibonacci's numbers. More specifically, the equation $\mathbf{u}_{n+1} = A\mathbf{u}_n$ for $n \geq 0$ encodes the relation $f_{n+2} = f_{n+1} + f_n$ (the second relation encoded is the tautology $f_{n+1} = f_{n+1}$).

Notice that (2) gives us a way to calculate $\mathbf{u}_n$ (that is f_n and f_{n+1}) from $\mathbf{u}_0$ (that is f_0 and f_1) by means of the equation $\mathbf{u}_n = A^n \mathbf{u}_0$.

Let us compute the eigenvalues and eigenvectors of the matrix A introduced above. Despite the fact that A is rather simple, the eigenvalues and the eigenvectors of A are not 'nice' at all! The characteristic polynomial of A is

$$\det \left[\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

Hence we obtain the following eigenvectors

$$\lambda^2 - \lambda - 1 = 0 \qquad \Longleftrightarrow \qquad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

$\boxed{\lambda_1 = \dfrac{1+\sqrt{5}}{2} \textbf{:}}$ In order to find (one of) the eigenvector(s) $\mathbf{v}_1$ associated to λ_1 we need to solve the following system of equations

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \qquad \Longleftrightarrow \qquad (A - \lambda_1 I_2)\mathbf{v}_1 = \mathbf{0}.$$

That is we need to find the solutions of

$$\underbrace{\begin{bmatrix} 1 - \dfrac{1+\sqrt{5}}{2} & 1 \\ 1 & -\dfrac{1+\sqrt{5}}{2} \end{bmatrix}}_{A - \lambda_1 I_2} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\mathbf{v}_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since we are subtracting one of the two values that make the matrix A singular, we have that the system of two linear equations in a and b reduces to the single equation

$$a - \frac{1+\sqrt{5}}{2} b = 0.$$

If we set $b = 1$ then $a = \dfrac{1+\sqrt{5}}{2}$. Hence we get the eigen pair

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad\qquad \mathbf{v}_1 = \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

$\boxed{\lambda_2 = \dfrac{1-\sqrt{5}}{2} \textbf{:}}$ In order to find (one of) the eigenvector(s) $\mathbf{v}_2$ associated to λ_2 we need to solve the following system of equations

$$A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \qquad \Longleftrightarrow \qquad (A - \lambda_2 I_2)\mathbf{v}_2 = \mathbf{0}.$$

That is we need to find the solutions of

$$\underbrace{\begin{bmatrix} 1 - \dfrac{1-\sqrt{5}}{2} & 1 \\ 1 & -\dfrac{1-\sqrt{5}}{2} \end{bmatrix}}_{A - \lambda_2 I_2} \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{\mathbf{v}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since we are subtracting the other of the two values that make the matrix A singular, we have that the system of two linear equations in a and b reduces to the single equation

$$c - \frac{1-\sqrt{5}}{2} d = 0.$$

If we set $d = 1$ then $c = \dfrac{1-\sqrt{5}}{2}$. Hence we get the eigen pair

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \qquad\qquad \mathbf{v}_2 = \begin{bmatrix} \dfrac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

Let us now rewrite the vector $\mathbf{u}_0$ as a linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. That is we are seeking values c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{u}_0$

$$c_1 \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \dfrac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Longleftrightarrow \qquad \begin{bmatrix} c_1 \dfrac{1+\sqrt{5}}{2} + c_2 \dfrac{1-\sqrt{5}}{2} \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\iff \qquad \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} & \dfrac{1-\sqrt{5}}{2} \\ & \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This system of linear equations leads to the solutions $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$. That is

$$\mathbf{u}_0 = \frac{1}{\sqrt{5}}\,\mathbf{v}_1 - \frac{1}{\sqrt{5}}\,\mathbf{v}_2.$$

Hence the relation $\mathbf{u}_n = A^n\mathbf{u}_0$ translates into the following

$$\mathbf{u}_n = A^n\left(\frac{1}{\sqrt{5}}\,\mathbf{v}_1 - \frac{1}{\sqrt{5}}\,\mathbf{v}_2\right) \iff \mathbf{u}_n = \frac{1}{\sqrt{5}}\,A^n\mathbf{v}_1 - \frac{1}{\sqrt{5}}\,A^n\mathbf{v}_2 \iff \mathbf{u}_n = \frac{1}{\sqrt{5}}\,\lambda_1^n\mathbf{v}_1 - \frac{1}{\sqrt{5}}\,\lambda_2^n\mathbf{v}_2$$

$$\iff \quad \begin{bmatrix} f_{n+1} \\ \\ f_n \end{bmatrix} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} \\ \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \begin{bmatrix} \dfrac{1-\sqrt{5}}{2} \\ \\ 1 \end{bmatrix}$$

(In other words we have $\mathbf{u}_n = A^n\mathbf{u}_0 = A^n(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A^n\mathbf{v}_1 + c_2A^n\mathbf{v}_2 = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2$.)

The above matrix equation translates into the following two (consistent) expressions

$$f_{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \qquad \text{and} \qquad \boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.}$$

Observe that the eigenvalue $\lambda_1 = (1+\sqrt{5})/2 \approx 1.618$ (the largest, or *dominant*, eigenvalue) and the eigenvalue $\lambda_2 = (1-\sqrt{5})/2 \approx -0.618$. Hence $\lambda_2^n \to 0$ (in an oscillatory fashion) as $n \to \infty$. Thus we conclude that (for n sufficiently large)

$$\boxed{f_n = \text{closest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.}$$

The eigenvalue $\lambda_1 = (1+\sqrt{5})/2$ is also called the *golden ratio*.

Let us check the above result in the chart below

n	$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$	f_n
0	0.4472	0
1	0.7235	1
2	1.1707	1
3	1.8943	2
4	3.0649	3
5	4.9591	5
6	8.0239	8
7	12.9826	13
8	21.0059	21
9	33.9876	34
$\vdots$	$\vdots$	$\vdots$

A population model. If we analyze our description of the Fibonacci's numbers, we realize that we had a matrix A that was giving us a *transition* between a *set/state* $\mathbf{u}_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ to the next set/state $\mathbf{u}_{n+1} = \begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix}$. Thus, what we did for the Fibonacci's numbers can be applied to describe the dynamics of a population that have a finite number (not necessarily two) of stages in life.

Suppose that we consider an hypothetical animal that has two life stages:

juvenile adult.

Suppose that we count the number of members of this population on a weekly basis. Say

$J_t =$ number of juveniles at week t $A_t =$ number of adults at week t.

The relation between the population during two consecutive weeks can reasonably be described as follows

$$\boxed{J_{t+1} = J_t - mJ_t - gJ_t + fA_t,} \tag{3}$$

where the term "$-mJ_t$" accounts for the fraction of juveniles that dies, the term "$-gJ_t$" accounts for the fraction of juveniles that becomes adult, and the term "$+fA_t$" accounts for the newborns;

$$\boxed{A_{t+1} = A_t - \mu A_t + gJ_t,} \tag{4}$$

where the term "$-\mu A_t$" accounts for the fraction of adults that dies and the term "$+gJ_t$" accounts for the fraction of juveniles that becomes adult.

Observe that m, g, f, μ are numbers that denote weekly rates, which we assume to be constant for each period. We can express express the relations described above in matrix form as

$$\underbrace{\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix}}_{\text{state } t+1} = \underbrace{\begin{bmatrix} 1-m-g & f \\ g & 1-\mu \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} J_t \\ A_t \end{bmatrix}}_{\text{state } t}. \tag{5}$$

If we define the vector $\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix}$ for any integer $t \geq 0$ then we can rewrite the above expression in the following recursive way

$$\mathbf{u}_{t+1} = A\mathbf{u}_t \qquad \text{with} \quad \mathbf{u}_0 = \begin{bmatrix} J_0 \\ A_0 \end{bmatrix};$$

in an explicit form we have

$$\mathbf{u}_t = A^t\mathbf{u}_0 \qquad \text{with} \quad \mathbf{u}_0 = \begin{bmatrix} J_0 \\ A_0 \end{bmatrix}.$$

Since we are interested in the dynamics of this population we have another example of large powers of a matrix.

Example: We illustrate the above model with a numerical example. Suppose that $g = m = 0.5$, $f = 2$, and $\mu = 0.9$. Hence (5) becomes

$$\mathbf{u}_{t+1} = \begin{bmatrix} 0 & 2 \\ 0.5 & 0.1 \end{bmatrix} \mathbf{u}_t.$$

We can easily check that the eigen pairs are

$$\lambda_1 = 1.051 \quad \leftrightsquigarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1.9029 \\ 1 \end{bmatrix} \qquad\qquad \lambda_2 = -0.951 \quad \leftrightsquigarrow \quad \mathbf{v}_2 = \begin{bmatrix} -2.102 \\ 1 \end{bmatrix}.$$

As we discussed in the case of the Fibonacci's numbers, the general solution to our problem is

$$\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix} = A^t\mathbf{u}_0 = A^t(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A^t\mathbf{v}_1 + c_2A^t\mathbf{v}_2 = c_1\lambda_1^t\mathbf{v}_1 + c_2\lambda_2^t\mathbf{v}_2,$$

where c_1 and c_2 are the values that allow us to rewrite the vector $\mathbf{u}_0$ as a linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
As it happened in the case of the Fibonacci's numbers, one of the eigenvalues is dominant. Namely, $\lambda_1 = 1.051 > -0.951 = \lambda_2$. Thus we can rewrite our solution as

$$\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix} = \lambda_1^t \left(c_1 \mathbf{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^t \mathbf{v}_2 \right) \underset{\text{as } t \to \infty}{\approx} \lambda_1^t c_1 \mathbf{v}_1.$$

It makes sense to call the *dominant* eigenvalue λ_1 the *growth rate* ($\lambda_1 = 1.051 \leftrightsquigarrow$ growth rate $= 5.1\%$) and the corresponding eigenvector $\mathbf{v}_1$ the *stable age structure*. Also observe that the second term in the general solution leads to an oscillating (decaying) behavior caused by the factor $(-0.951)^t$.

As we observed earlier in the long run we have

$$\mathbf{u}_t = \begin{bmatrix} J_t \\ A_t \end{bmatrix} \approx c_1 (1.051)^t \begin{bmatrix} 1.9029 \\ 1 \end{bmatrix}.$$

This implies that the ratio $\dfrac{J_t}{A_t} = \dfrac{c_1(1.051)^t 1.9029}{c_1(1.051)^t} = 1.9029$ is constant. This means that in the long run the population will consist of 65.6% of juveniles and 34.4% of adults[1]. In other words there will be about 1.9 juveniles for every adult.

Remark. The above population model is an example of a *Leslie matrix*. You can read more about Leslie matrices (even for populations with more than two life stages!) on pages 459-464 and 483-486 of our textbook *Calculus for Biology and Medicine* by Claudia Neuhauser. The example is taken from the book *Mathematical Methods in Biology* by J.D. Logan and W. Wolensky (pages 103-105).

References

[1] E. Batschelet, *Introduction to Mathematics for Life Scientists.*
[2] L. Edelstein-Keshet, *Mathematical Models in Biology.*
[3] D.E. Goldberg and J.H. Vandermeer, *Population Ecology.*
[4] J.D. Logan and W. Wolesensky, *Mathematical Methods in Biology.*
[5] J.D. Murray, *Mathematical Biology.*
[6] C. Neuhauser, *Calculus for Biology and Medicine.*
[7] C. H. Taubes, *Modeling Differential Equations in Biology.*

[1]If x represents the percentage of juvenile then $100 - x$ represents the percentage of adults. Hence the equation (ratio) $x/(100 - x) = 1.9029$ gives the solution $x = 190.29/2.9029 \approx 65.6$ and $100 - x \approx 34.4$.