MA 138 – Calculus 2 with Life Science Applications

Section 6.3 (Applications of integration)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Section 6.3: Applications of Integration

We are interested in the following three applications of integrals:

(1) **average** of a continuous function on $[a, b]$;

(2) **area between curves**;

(3) **cumulative change**.

Average Values

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{avg}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

In general, let's try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $c_1, \ldots, c_n$ in successive subintervals and calculate the average of the numbers $f(c_1), \ldots, f(c_n)$:

$$\frac{f(c_1) + \cdots + f(c_n)}{n}$$

Since $\Delta x = (b-a)/n$, we can write $1/n = \Delta x/(b-a)$ and the average value becomes

$$\frac{f(c_1)\Delta x + \cdots + f(c_n)\Delta x}{b-a} = \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x.$$

If we let n increase, we would be computing the average value of a large number of closely spaced values. More precisely,

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Average of a Continuous Function on $[a, b]$

Assume that $f(x)$ is a continuous function on $[a, b]$. The average value of f on the interval $[a, b]$ is defined to be

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

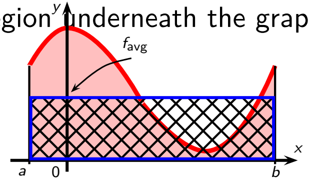
Geometric Meaning

Mean Value Theorem for Definite Integrals

Assume that $f(x)$ is a continuous function on $[a, b]$. Then there exists a number $c \in [a, b]$ such that

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

That is, when f is continuous, there exists a number c such that $f(c) = f_{avg}$. If f is a continuous, positive valued function, f_{avg} is that number such that the rectangle with base $[a, b]$ and height f_{avg} has the same area as the region underneath the graph of f from a to b.



Example 1 (Online Homework #14)

If a cup of coffee has temperature 95°C in a room where the temperature is 20°C, then, according to Newton's Law of Cooling, the temperature of the coffee after t minutes is

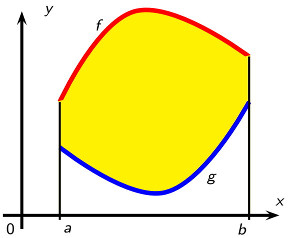
$$T(t) = 20 + 75e^{-t/50}.$$

What is the average temperature (in degrees Celsius) of the coffee during the first half hour?

Area Between Curves

Assume f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$. The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$



Example 2 (Online Homework #2)

Find the area of the region enclosed by the two functions $y = 7x^2$ and $y = x^2 + 6$.

Example 3 (Online Homework #3)

Find the area between $y = 8\sin x$ and $y = 10\cos x$ over the interval $[0, \pi]$. Sketch the curves if necessary.

Example 4 (Online Homework #4)

Find the area between $y = e^x$ and $y = e^{4x}$ over $[0, 1]$.

Example 5 (Online Homework #6)

Find the area of the quadrangle with vertices $(4, 2)$, $(-5, 4)$, $(-2, -4)$, and $(3, -3)$.

Example 6 (Online Homework #7)

Consider the area between the graphs $x + y = 14$ and $x + 6 = y^2$.

This area can be computed in two different ways using integrals.

- First of all it can be computed as a sum of two integrals

$$\int_a^b f(x)\,dx + \int_b^c g(x)\,dx$$

  where $a =$ ____, $b =$ ____, $c =$ ____, and $f(x) =$ ____ $g(x) =$ ____.

- Alternatively this area can be computed as a single integral

$$\int_\alpha^\beta h(y)\,dy$$

  where $\alpha =$ ____, $\beta =$ ____, and $h(y) =$ ____.

Example 7 (Online Homework #5)

Find the value(s) of c such that the area of the region bounded by the parabolae $y = x^2 - c^2$ and $y = c^2 - x^2$ is 1944.

Cumulative Change

Suppose that we have a population whose size at time t is given by $N(t)$. Suppose further that its rate of growth is given by the initial value problem

IVP: $$\frac{dN}{dt} = f(t) \qquad N(0) = N_0.$$

Then, by Part I of the Fundamental Theorem of Calculus we have that

$$N(t) = \int_0^t f(u)\,du + C$$

represents all antiderivatives of $f(t)$ [or dN/dt].

Now, $N(0) = \underbrace{\int_0^0 f(u)\,du}_{=0} + C = C$ so $C = N_0 = N(0)$. Therefore

$$N(t) = \int_0^t f(u)\,du + N_0 \quad \text{or} \quad \boxed{N(t) - N(0) = \int_0^t f(u)\,du}.$$

More generally, the IVP: $\dfrac{dN}{dt} = f(t) \qquad N(a) = N_a$ has solution

$$\boxed{N(t) - N(a) = \int_a^t f(u)\,du = \int_a^t \frac{dN}{du}\,du}.$$

That is

$$\left\{ \begin{array}{c} \text{cumulative change} \\ \text{on the interval } [a, t] \end{array} \right\} = \int_a^t \left\{ \begin{array}{c} \text{instantaneous rate of} \\ \text{change at time } u \end{array} \right\} du$$

Similarly, if $p(t)$ is the position function of an object at time t, then

$$\frac{dp}{dt} = v(t) \qquad\qquad p(a) = p_a$$

gives $\rightsquigarrow$ $$\boxed{\underbrace{p(b) - p(a)}_{\text{distance traveled on } [a,b]} = \int_a^b v(t)\,dt = \int_a^b \frac{dp}{dt}\,dt}.$$

Example 8 (Problem #2, Section 6.3, page 349)

Suppose the change in biomass $B(t)$ at time t during the interval $[0, 12]$ follows the equation

$$\frac{dB}{dt} = \cos\left(\frac{\pi}{6}t\right).$$

How does the biomass at time $t = 12$ compare to the biomass at time $t = 0$?

Example 9 (Problem #6, Section 6.3, page 349)

If $\frac{dw}{dx}$ represents the rate of change of the weight of an organism of age x, explain what

$$\int_3^5 \frac{dw}{dx}\,dx$$

means.