MA 138 – Calculus 2 with Life Science Applications
Rational Functions and Partial Fractions
(Section 7.3)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Example 1

Evaluate the following indefinite integrals

- $$\int \frac{5}{(3x+2)^4}\,dx;$$

- $$\int \frac{2x-2}{(x^2-2x+5)^3}\,dx.$$

Section 7.3: Rational Functions and Partial Fractions

- A rational function f is the quotient of two polynomials. That is,
$$f(x) = \frac{P(x)}{Q(x)}$$
where $P(x)$ and $Q(x)$ are polynomials.
- To integrate such a function, we write $f(x)$ as a sum of a polynomial and simpler rational functions (=partial-fraction decomposition).
- These simpler rational functions, which can be integrated with the methods we have learned, are of the form
$$\frac{A}{(ax+b)^n} \qquad \text{or} \qquad \frac{Bx+C}{(ax^2+bx+c)^n}$$
where A, B, C, a, b, and c are constants and n is a positive integer.
- In this form, the quadratic polynomial ax^2+bx+c can no longer be factored into a product of two linear functions with real coefficients.

Proper Rational Functions

- The rational function $f(x) = P(x)/Q(x)$ is said to be **proper** if the degree of the polynomial in the numerator, $P(x)$, is strictly less than the degree of the polynomial in the denominator, $Q(x)$,

$$f(x) = \frac{P(x)}{Q(x)} \quad \textbf{proper} \quad \iff \quad \deg P(x) < \deg Q(x).$$

- Which of the following three rational functions

$$f_1(x) = \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} \qquad f_2(x) = \frac{x}{x+2} \qquad f_3(x) = \frac{2x-3}{x^2+x}$$

is proper? Only $f_3(x)$ is proper.

- The first step in the partial-fraction decomposition procedure is to use the long division algorithm to write $f(x)$ as a sum of a polynomial and a **proper** rational function.

Algebra Review

Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials*.

Long Division Algorithm

If $A(x)$ and $B(x)$ are polynomials, with $B(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree strictly less than the degree of $B(x)$, such that

$$A(x) = Q(x) \cdot B(x) + R(x)$$

The polynomials $A(x)$ and $B(x)$ are called the **dividend** and **divisor**, respectively; $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

Example 2

Divide the polynomial

$$A(x) = 2x^2 - x - 4 \quad \text{by} \quad B(x) = x - 3.$$

$$\begin{array}{r|rrrr} & 2x & & & \\ \cline{2-5} x - 3 & 2x^2 & - & x & - 4 \\ & 2x^2 & - & 6x & \\ \cline{2-4} & & + & 5x & - 4 \end{array}$$

(Complete the above table and check your work!)

- **Synthetic division** is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$, where c is a number. In synthetic division we write only the essential part of the long division table.
- In synthetic division we abbreviate the polynomial $A(x)$ by writing only its coefficients.
  Moreover, instead of $B(x) = x - c$, we simply write 'c.'
  Writing c instead of $-c$ allows us to add instead of subtract!

Example 2 (revisited): Divide

$$A(x) = 2x^2 - x - 4 \text{ by } B(x) = x - 3.$$

3	2	−1	−4
		6	15
	2	5	11

We obtain $Q(x) = 2x + 5$ and $R(x) = 11$. That is,

$$2x^2 - x - 4 = (2x + 5)(x - 3) + 11.$$

Example 3 (Online Homework # 3)

Use the Long Division Algorithm to write $f(x)$ as a sum of a polynomial and a proper rational function

$$f(x) = \frac{x^3}{x^2 + 4x + 3}.$$

Partial Fraction Decomposition (linear factors)

Case of Distinct Linear Factors

$Q(x)$ is a product of m distinct linear factors. $Q(x)$ is thus of the form

$$Q(x) = a(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_m)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the m distinct roots of $Q(x)$.
The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_m}{x - \alpha_m}\right]$$

We will see in the next examples how the constants $A_1, A_2, \ldots, A_m$ are determined.

Example 3 (cont.d)

Evaluate the indefinite integral: $\int \frac{x^3}{x^2+4x+3}\,dx.$

[**Note:** from the calculations carried out in the first part of the example, we know that our problem reduces to

$$\int (x-4)\,dx + \int \frac{13x+12}{(x+3)(x+1)}\,dx.$$

]

(Heaviside) cover-up method

We illustrate this method by using the previous example:

$$\frac{13x+12}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$$

$$\leftrightsquigarrow$$

$$\boxed{A(x+1)+B(x+3) = 13x+12} \qquad (*)$$

Set $x=-1$ in $(*)$. We obtain

$$A\cdot 0 + B\cdot(-1+3) = 13(-1)+12$$

$$\leftrightsquigarrow$$

$$B\cdot(2) = -1$$

$$\leftrightsquigarrow$$

$$B = -1/2$$

Set $x=-3$ in $(*)$. We obtain

$$A\cdot(-3+1) + 0 = 13(-3)+12$$

$$\leftrightsquigarrow$$

$$A\cdot(-2) = -27$$

$$\leftrightsquigarrow$$

$$A = 27/2$$

Example 4 (Online Homework # 8)

Find the integral: $\displaystyle\int_2^5 \frac{2}{x^2-1}\,dx.$

Example 5 (Online Homework # 6)

Evaluate the indefinite integral: $\displaystyle\int \frac{1}{x(x+1)}\,dx$.