MA 138 – Calculus 2 with Life Science Applications Rational Functions and Partial Fractions (Section 7.3)

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Lecture 6

Example 1

Evaluate the following indefinite integrals

•
$$\int \frac{5}{(3x+2)^4} dx;$$

• $\int \frac{2x-2}{(x^2-2x+5)^3} dx.$

Section 7.3: Rational Functions and Partial Fractions

A rational function *f* is the quotient of two polynomials. That is,

$$f(x) = \frac{P(x)}{Q(x)}$$

where P(x) and Q(x) are polynomials.

- To integrate such a function, we write f(x) as a sum of a polynomial and simpler rational functions (=partial-fraction decomposition).
- These simpler rational functions, which can be integrated with the methods we have learned, are of the form

$$\frac{A}{(ax+b)^n}$$
 or $\frac{Bx+C}{(ax^2+bx+c)^n}$

where A, B, C, a, b, and c are constants and n is a positive integer.

In this form, the quadratic polynomial $ax^2 + bx + c$ can no longer be factored into a product of two linear functions with real coefficients.

Proper Rational Functions

• The rational function f(x) = P(x)/Q(x) is said to be **proper** if the degree of the polynomial in the numerator, P(x), is strictly less than the degree of the polynomial in the denominator, Q(x),

$$f(x) = rac{P(x)}{Q(x)}$$
 proper \iff deg $P(x) <$ deg $Q(x)$.

Which of the following three rational functions

$$f_1(x) = \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} \qquad f_2(x) = \frac{x}{x + 2} \qquad f_3(x) = \frac{2x - 3}{x^2 + x}$$

is proper? Only $f_3(x)$ is proper.

The first step in the partial-fraction decomposition procedure is to use the long division algorithm to write f(x) as a sum of a polynomial and a proper rational function. Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials*.

Long Division Algorithm

If A(x) and B(x) are polynomials, with $B(x) \neq 0$, then there exist unique polynomials Q(x) and R(x), where R(x) is either 0 or of degree strictly less than the degree of B(x), such that

$$A(x) = Q(x) \cdot B(x) + R(x)$$

The polynomials A(x) and B(x) are called the **dividend** and **divisor**, respectively; Q(x) is the **quotient** and R(x) is the **remainder**.

Example 2

Divide the polynomial

$$A(x) = 2x^2 - x - 4$$
 by $B(x) = x - 3$.

$$x - 3 \overline{\smash{\big)}\ 2x^2 - x - 4}$$

$$\underline{2x^2 - 6x}$$

$$+ 5x - 4$$

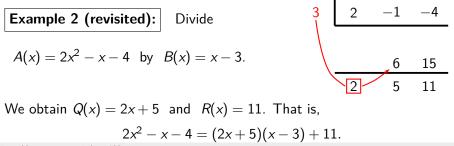
(Complete the above table and check your work!)

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- **Synthetic division** is a quick method of dividing polynomials; it can be used when the divisor is of the form *x* − *c*, where *c* is a number. In synthetic division we write only the essential part of the long division table.
- In synthetic division we abbreviate the polynomial A(x) by writing only its coefficients.

Moreover, instead of B(x) = x - c, we simply write 'c.' Writing c instead of -c allows us to add instead of subtract!



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Lecture 6

Example 3 (Online Homework # 3)

Use the Long Division Algorithm to write f(x) as a sum of a polynomial and a proper rational function

$$f(x) = \frac{x^3}{x^2 + 4x + 3}$$

Partial Fraction Decomposition (linear factors)

Case of Distinct Linear Factors

Q(x) is a product of *m* distinct linear factors. Q(x) is thus of the form

$$Q(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_m)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the *m* distinct roots of Q(x). The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a} \left[\frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \dots + \frac{A_m}{x - \alpha_m} \right]$$

We will see in the next examples how the constants A_1, A_2, \ldots, A_m are determined.

Example 3 (cont.d)

Evaluate the indefinite integral:

$$\int \frac{x^3}{x^2 + 4x + 3} \, dx$$

Note: from the calculations carried out in the first part of the example, we know that our problem reduces to

$$\int (x-4) \, dx + \int \frac{13x+12}{(x+3)(x+1)} \, dx.$$

(Heaviside) cover-up method

We illustrate this method by using the previous example:

$$\frac{13x+12}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1) + B(x+3)}{(x+3)(x+1)}$$

$$A(x+1) + B(x+3) = 13x + 12$$
 (*)

Set
$$x = -1$$
 in (*). We obtain
 $A \cdot 0 + B \cdot (-1 + 3) = 13(-1) + 12$
 $B \cdot (2) = -1$
 $B = -1/2$
Set $x = -3$ in (*). We obtain
 $A \cdot (-3 + 1) + 0 = 13(-3) + 12$
 $A \cdot (-2) = -27$
 $A = 27/2$

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Example 4 (Online Homework # 8)

Find the integral:

$$\int_{2}^{5} \frac{2}{x^2 - 1} \, dx.$$

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Example 5 (Online Homework # 6)

Evaluate the indefinite integral:

$$\int \frac{1}{x(x+1)} \, dx.$$

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