MA 138 – Calculus 2 with Life Science Applications
Rational Functions and Partial Fractions
(Section 7.3)

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Example 1 (Online Homework # 7)

Evaluate the indefinite integral: $\displaystyle\int \frac{3}{(x+a)(x+b)}\,dx.$

Example 2

Consider the rational function

$$f(x) = \frac{4x^2 - x - 1}{(x+1)^2(x-3)}$$

which has a repeated factor at the denominator.

Try to find constants A and B such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)^2} + \frac{B}{(x-3)}.$$

Example 2 (again)

The previous calculation didn't work.

Try now to find constants A, B and C such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-3)}.$$

Then evaluate the definite integral

$$\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)}\, dx.$$

Partial Fraction Decomposition
(repeated linear factors)

Case of Repeated Linear Factors

$Q(x)$ is a product of m distinct linear factors to various powers. $Q(x)$ is thus of the form

$$Q(x) = a(x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2}\cdots(x-\alpha_m)^{n_m}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the m distinct roots of $Q(x)$ and $n_1, n_2, \ldots, n_m$ are positive integers such that $n_1 + n_2 + \cdots + n_m = \deg Q(x)$.
The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\sum_{i=1}^{m} \frac{A_{i,1}}{x-\alpha_i} + \frac{A_{i,2}}{(x-\alpha_i)^2} + \cdots + \frac{A_{i,n_i}}{(x-\alpha_i)^{n_i}}\right].$$

Example 3 (Online Homework # 5)

Evaluate the integral

$$\int \frac{-10x^2}{(x+1)^3}\,dx.$$

Example 4

Evaluate the integral

$$\int \frac{1}{x^2(x-1)^2}\,dx.$$

Example 5 (Online Homework #15)

If $f(x)$ is a quadratic function such that $f(0) = 1$ and $\displaystyle\int \frac{f(x)}{x^2(x+1)^3}\,dx$ is a rational function, find the value of $f'(0)$.

Partial Fraction Decomposition
(irreducible quadratic factors)

Irreducible quadratic factors in the denominator of a proper rational functions are dealt with in the partial-fraction decomposition as follows:

Case of Irreducible Quadratic Factors

If the irreducible quadratic factor $ax^2 + bx + c$ is contained n times in the factorization of the denominator of a proper rational function, then the partial-fraction decomposition contains terms of the form

$$\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_n x + C_n}{(ax^2 + bx + c)^n}.$$

Example 6 (Example 11, Section 7.3, page 384)

Write the partial faction decomposition of

$$f(x) = \frac{2x^3 - x^2 + 2x - 2}{(x^2+1)(x^2+2)}.$$