MA 138 – Calculus 2 with Life Science Applications
Improper Integrals
(Section 7.4)

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Improper Integrals

We discuss definite integrals of two types with the following characteristics:

(1) **one or both limits of integration are infinite**; that is, the integration interval is unbounded. For example

$$\int_1^\infty e^{-x}\,dx \qquad \text{or} \qquad \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx;$$

(These integrals are very important in Probability and Statistics!)

(2) **the integrand becomes infinite at one or more points of the interval of integration**. For example

$$\int_{-1}^1 \frac{1}{x^2}\,dx \qquad \text{or} \qquad \int_0^1 \frac{1}{2\sqrt{x}}\,dx.$$

We call such integrals **improper integrals**.

Type 2: Unbounded Integrand

What if the integrand becomes infinite at one or both endpoints of the interval of integration?

- If f is continuous on $(a, b]$ and $\lim_{x \longrightarrow a^+} f(x) = \pm\infty$, we define
$$\int_a^b f(x)\,dx := \lim_{c \longrightarrow a^+} \int_c^b f(x)\,dx$$
provided that this limit exists.
- If f is continuous on $[a, b)$ and $\lim_{x \longrightarrow b^-} f(x) = \pm\infty$, we define
$$\int_a^b f(x)\,dx := \lim_{c \longrightarrow b^-} \int_a^c f(x)\,dx$$
provided that this limit exists.

If the limit does not exist, we say that the integral diverges.

Example 1

Determine whether the improper integral

$$\int_{1}^{e} \frac{1}{x\sqrt{\ln x}}\, dx$$

is convergent. If the integral is convergent, compute its value.

Example 2 (Problem #28, Section 7.4, page 396)

Determine whether the improper integral

$$\int_1^e \frac{1}{x \ln x}\, dx$$

is convergent. If the integral is convergent, compute its value.

Example 3 (Online Homework #7)

Determine whether the improper integral

$$\int_0^9 \frac{4}{(x-6)^2}\,dx$$

is convergent. If the integral is convergent, compute its value.

Example 4 (Problem #36, Section 7.4, page 397)

Let p be a positive real number. Show that

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p} & \text{for } 0 < p < 1 \\ \infty & \text{for } p \geq 1. \end{cases}$$

E.g.: $\int_0^1 \frac{1}{x}\,dx$ and $\int_0^1 \frac{1}{x^2}\,dx$ both diverge (as $p = 1, 2$, respectively).

E.g.: $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$ and $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}$ (as $p = 1/2, 1/3$, respectively).

Example 5 (Problem #15, Section 7.4, page 396)

Determine whether the improper integral

$$\int_{-1}^{1} \ln |x| \, dx.$$

is convergent. If the integral is convergent, compute its value.