MA 138 – Calculus 2 with Life Science Applications
Improper Integrals
(Section 7.4)

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A Comparison Result for Improper Integrals

In many cases, it is difficult (if not impossible) to evaluate an integral exactly. For example, it takes some work to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi.$$

In dealing with improper integrals, it frequently suffices to know whether the integral converges.

Instead of computing the value of the improper integral exactly, we can then resort to simpler integrals that either dominate or are dominated by the improper integral of interest.

We will explain this idea graphically.

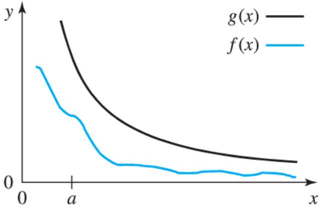
Convergence Test

Test for Convergence

We assume that $f(x) \geq 0$ for $x \geq a$.

To show that $\int_a^\infty f(x)\,dx$ is convergent it is enough to find a function $g(x)$ such that

- $g(x) \geq f(x)$ for all $x \geq a$;
- $\int_a^\infty g(x)\,dx$ is convergent.

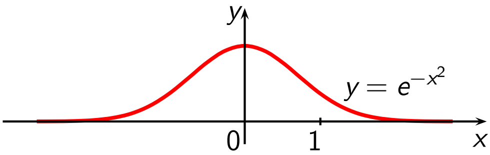


It is clear from the graph that $0 \leq \int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$

If $\int_a^\infty g(x)dx < \infty$, it follows that $\int_a^\infty f(x)\,dx$ is convergent, since $\int_a^\infty f(x)\,dx$ must take on a value between 0 and $\int_a^\infty g(x)dx$.

Example 1 (Example #9, Section 7.4, p. 395)

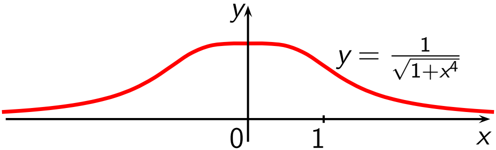
Show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.



Note: It is an hard fact to show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Example 2 (Problem #38, Section 7.4, p. 397)

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^4}}\,dx$ converges.



Example 3 (Online Homework # 8)

Let f(x) be a continuous function defined on the interval $[2, \infty)$ such that

$$f(4) = 7 \qquad |f(x)| < x^3 + 3 \qquad \int_4^{\infty} f(x)e^{-x/8}\,dx = -6.$$

Determine the value of

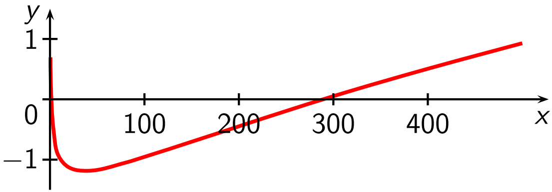
$$\int_4^{\infty} f'(x)e^{-x/8}\,dx.$$

Example 4 (Problem #8(b), Exam 1, Spring 14)

It is given that for $x \geq 300$ the inequality $3 \ln x \leq \sqrt{x}$ holds. Use the above inequality and the Comparison Theorem for improper integrals to conclude that

$$\int_{300}^{\infty} e^{-\sqrt{x}}\, dx$$

coverges.



graph of $y = \sqrt{x} - 3\ln(x)$

Example 5 (Problem #9(b), Exam 1, Spring 13)

If x is really big, say bigger than 100, one has

$$-\sqrt{x} \leq -2\ln x = \ln\left(\frac{1}{x^2}\right).$$

Use this inequality together with the fact that the exponential is an increasing function to determine if

$$\int_{100}^{\infty} e^{-\sqrt{x}}\,dx$$

converges or diverges.

Example 6 (Problem #44, Section 7.4, p. 397)

Determine whether $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx$ is convergent or not.

Example 7

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx \leq 4.$

Note: We will show at the end of the lecture that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi.$

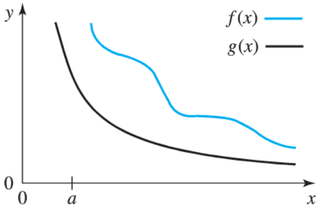
Divergence Test

Test for Divergence

We assume that $f(x) \geq 0$ for $x \geq a$.

To show that $\int_a^\infty f(x)\,dx$ is divergent it is enough to find a function $g(x)$ such that

- $g(x) \leq f(x)$ for all $x \geq a$;
- $\int_a^\infty g(x)\,dx$ is divergent.



It is clear from the graph that $\int_a^\infty f(x)\,dx \geq \int_a^\infty g(x)\,dx \geq 0$.

If $\int_a^\infty g(x)dx$ is divergent, it follows that $\int_a^\infty f(x)\,dx$ is divergent.

Example 8 (Example #10, Section 7.4, p. 396)

Show that $\displaystyle\int_1^\infty \frac{1}{\sqrt{x+\sqrt{x}}}\,dx$ is divergent.

The Inverse Tangent Function $\tan^{-1}(x)$

The only functions that have an inverse are one-to-one functions.

The tangent function is not one-to-one.

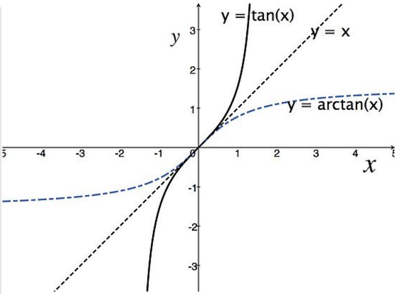
We can make it one-to-one by restricting its domain to the interval $(-\pi/2, \pi/2)$. Its inverse is denoted by $\tan^{-1}$ or arctan.

Inverse Tangent Function

The inverse tangent function, $\tan^{-1}$, has

- domain $\mathbb{R}$
- range $(-\pi/2, \pi/2)$.

$$\tan^{-1}(x) = y$$
$$\iff$$
$$x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$



Derivative of $\tan^{-1}(x)$ (Example 4, Section 4.10, p. 199)

We want to compute $\dfrac{d}{dx}(\tan^{-1}(x)) = \dfrac{dy}{dx}$.

Notice that $\tan^{-1}(x) = y$ is equivalent to $x = \tan(y)$. If we differentiate with respect to x the latter equation and apply the chain rule, we obtain

$$\frac{d}{dx}(x) = 1 = \frac{d}{dx}(\tan(y)) = \frac{d}{dy}(\tan(y)) \cdot \frac{dy}{dx} = \sec^2(y) \cdot \frac{dy}{dx}.$$

Thus

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1 + \tan^2(y)}.$$

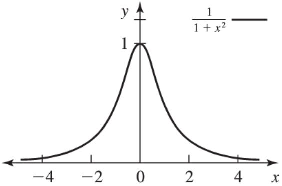
We used the trigonometric identity $\sec^2(y) = 1 + \tan^2(y)$ to get the denominator in the rightmost term. Since $x = \tan(y)$, it follows that $x^2 = \tan^2(y)$, and, hence,

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1 + x^2}.$$

Integral of $\frac{1}{1+x^2}$ (Example 4, Section 7.4, p. 391)

From the previous discussion we have

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C.$$



Moreover

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= 2\int_0^{\infty} \frac{1}{1+x^2}\,dx \\
&= 2\lim_{b\longrightarrow\infty} \int_0^{b} \frac{1}{1+x^2}\,dx \\
&= 2\lim_{b\longrightarrow\infty} [\tan^{-1}(x)]_0^b \\
&= 2\lim_{b\longrightarrow\infty} [\tan^{-1}(b) - \tan^{-1}(0)] \\
&= 2\,(\pi/2 - 0) = \pi.
\end{aligned}$$