MA 138 – Calculus 2 with Life Science Applications
Solving Differential Equations
(Section 8.1)

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Differential Equations ($\equiv$ DEs)

A **differential equation** is an equation that contains
an unknown function and one or more of its derivatives.

For example

- $\dfrac{dy}{dx} + 6y = 7;$
- $\dfrac{dy}{dt} + 0.2\,t\,y = 6t;$
- $\dfrac{dP}{dt} = \sqrt{P\,t};$
- $xy' + y = y^2.$

Differential equations can contain derivatives of any order; for example,

- $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = x\,y$ or $y'' + 6y' - x\,y = 0$

is a DE containing the first and second derivative of the function $y = y(x)$.

If a differential equation contains only the first derivative,
it is called a **first-order differential equation**: $\dfrac{dy}{dx} = h(x, y)$.

DEs arise for example in biology (e.g. models of population growth), economics (e.g. models of economic growth), and many other areas.

exponential growth model:

$$\frac{dN}{dt} = rN \qquad N(0) = N_0;$$

logistic growth model:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0;$$

von Bertalanffy models:

$$\frac{dL}{dt} = k(L_\infty - L) \qquad L(0) = L_0,$$

$$\frac{dW}{dt} = \eta W^{2/3} - \kappa W \qquad W(0) = W_0;$$

Solow's economic growth model:

$$\frac{dk}{dt} = sk^\alpha - \delta k \qquad k(0) = k_0.$$

Example 1

Consider the differential equation $$(t+1)\frac{dy}{dt} - y + 6 = 0.$$

Which of the following functions

$$y_1(t) = t + 7 \qquad y_2(t) = 3t + 21 \qquad y_3(t) = 3t + 9$$

are solutions for all t?

Separable Differential Equations

We will restrict ourselves to first-order differential equations

$$\frac{dy}{dx} = h(x, y) \qquad \text{of the form} \qquad \frac{dy}{dx} = f(x)g(y).$$

That is, the right-hand side of the equation is the product of two functions, one depending only on x, $f(x)$, the other only on y, $g(y)$.

Such equations are called **separable differential equations**.

This type of differential equations includes two special cases:

- **pure-time differential equations**: $\frac{dy}{dx} = f(x)$ [i.e., $g(y) \equiv 1$]
- **autonomous differential equations**: $\frac{dy}{dx} = g(y)$ [i.e., $f(x) \equiv 1$]

  (DEs of this form are frequently used in biological models.)

In order to solve the separable differential equation

$$\boxed{\frac{dy}{dx} = f(x)g(y)}, \qquad (*)$$

we divide both sides of $(*)$ by $g(y)$ [assuming that $g(y) \neq 0$]:

$$\frac{1}{g(y)}\,\frac{dy}{dx} = f(x).$$

Now, if $y = u(x)$ is a solution of $(*)$, then $u(x)$ satisfies

$$\frac{1}{g[u(x)]}u'(x) = f(x).$$

If we integrate both sides with respect to x, we find that

$$\int \frac{1}{g[u(x)]}u'(x)\,dx = \int f(x)\,dx \qquad \text{or} \qquad \boxed{\int \frac{1}{g(y)}\,dy = \int f(x)\,dx}$$

since $g[u(x)] = g(y)$ and $u'(x)dx = dy$.

Example 1 (again)

Solve the differential equation $(t+1)\frac{dy}{dt} - y + 6 = 0$.

Example 2 (Online Homework # 2)

Solve the following initial value problem

$$\frac{dy}{dt} + 0.2ty = 6t$$

with $y(0) = 4$.

Example 3 (Online Homework # 3)

Find the solution of the differential equation

$$\frac{dP}{dt} = \sqrt{P\,t}$$

that satisfies the initial condition $P(1) = 7$.

Example 4 (Online Homework # 5)

Find the solution of the differential equation

$$xy' + y = y^2$$

that satisfies the initial condition $y(1) = -1$.

Pure-Time Differential Equations

In many applications, the independent variable represents time. If the rate of change of a function depends only on time, we call the resulting differential equation a **pure-time differential equation**. Such a differential equation is of the form

$$\frac{dy}{dx} = f(x), \quad x \in I, \qquad y(x_0) = y_0,$$

where I is an interval and x represents time; the number x_0 is in the interval I.

The solution can then be written as

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(u)\, du.}$$

Example 5 (Example # 1, Section 8.1, p. 430)

Suppose that the volume $V(t)$ of a cell at time t changes according to

$$\frac{dV}{dt} = \sin t \qquad \text{with} \qquad V(0) = 3.$$

Find $V(t)$.

Autonomous Differential Equations

Many of the differential equations that model biological situations are of the form

$$\frac{dy}{dx} = g(y)$$

where the right-hand side does not explicitly depend on x. These equations are called **autonomous differential equations**.

Formally, we can solve this autonomous differential equation by separation of variables. We begin by dividing both sides of the equation by $g(y)$ and multiplying both sides by dx, to obtain

$$\frac{1}{g(y)}dy = dx.$$

Integrating both sides then gives $$\int \frac{1}{g(y)}dy = \int dx.$$

Example 6 (Online Homework # 1)

Find the particular solution of the differential equation

$$\frac{dy}{dx} + 6y = 7$$

satisfying the initial condition $y(0) = 0$.

Example 7 (Problem # 39, Section 8.1, p. 440)

Find the general solution of the differential equation $\frac{dy}{dx} = y^2 - 4.$