# MA 138 – Calculus 2 with Life Science Applications
# Matrices
## (Section 9.2)

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# Identity Matrix and Inverse of a Matrix

For any $n \geq 1$, the identity matrix is an $n \times n$ matrix, denoted by $I_n$, with 1's on its diagonal line and 0's elsewhere; that is,

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

## Property of the Identity Matrix

Suppose that $A$ is an $m \times n$ matrix. Then $I_m A = A = A I_n$.

## Inverse of a Matrix

Suppose that $A$ is an $n \times n$ square matrix. If there exists an $n \times n$ square matrix $B$ such that $AB = I_n = BA$ then $B$ is called the inverse matrix of $A$ and is denoted by $A^{-1}$.

# Example 1 (Part I)...Checking

Verify that:

- $A_1 = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and $B_1 = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix}$

  are inverses of each other. That is $A_1 B_1 = I_2 = B_1 A_1$.

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- $A_2 = \begin{bmatrix} 3 & 5 & -1 \\ 2 & -1 & 3 \\ 4 & 2 & -3 \end{bmatrix}$ and $B_2 = \frac{1}{73}\begin{bmatrix} -3 & 13 & 14 \\ 18 & -5 & -11 \\ 8 & 14 & -13 \end{bmatrix}$

  are inverses of each other. That is $A_2 B_2 = I_3 = B_2 A_2$.

# Matrix Representation of Linear Systems

We observe that the system of linear equations

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{cases}$$

can be written in matrix form as $AX = B$, where

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{B}$$

## The ... Guiding Light

- *A simple key observation*: To solve $5x = 10$ for $x$, we just divide both sides by 5 ($\equiv$ multiply both sides by $1/5 = 5^{-1}$). That is,
$$5x = 10 \quad \leftrightsquigarrow \quad 5^{-1} \cdot 5x = 5^{-1} \cdot 10 \quad \leftrightsquigarrow \quad x = 2$$
as $5^{-1} \cdot 5 = 1$ and $5^{-1} \cdot 10 = 2$.

- We have learnt how to write a system of $n$ linear equations in $n$ variables in the matrix form $AX = B$.

- To solve $AX = B$, we therefore need an operation that is *analogous to* multiplication by the 'reciprocal' of $A$. We have defined, whenever possible, a matrix $A^{-1}$ that serves this function (i.e., $A^{-1} \cdot A =$ Identity Matrix).

- Then, whenever possible, we can write the solution of $AX = B$ as
$$AX = B \quad \leftrightsquigarrow \quad A^{-1} \cdot AX = A^{-1} \cdot B \quad \leftrightsquigarrow \quad X = A^{-1} \cdot B.$$

## Example 1 (Part II)

Using the results verified in Example 1 (Part I) and our *Guiding Light* ($\equiv$ Principle), solve the following systems of linear equations by transforming them into matrix form

- $$\begin{cases} 3x + 5y = 7 \\ 2x + 4y = 6 \end{cases}$$

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- $$\begin{cases} 3x + 5y - z = 10 \\ 2x - y + 3z = 9 \\ 4x + 2y - 3z = -1 \end{cases}$$

# Properties of Matrix Inverses

The following properties of matrix inverses are often useful.

## Properties of Matrix Inverses

Suppose $A$ and $B$ are both invertible $n \times n$ matrices then

- $A^{-1}$ is unique;
- $(A^{-1})^{-1} = A$;
- $(AB)^{-1} = B^{-1}A^{-1}$;
- $(A^T)^{-1} = (A^{-1})^T$.

# How do we find the inverse (if possible) of a matrix?

- First of all the matrix has to be a square matrix!
- Suppose $n = 2$. For example, $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$.
- We need to find a matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AB = I_2 = BA$.
- $AB = I_2 \quad \leftrightsquigarrow \quad \begin{bmatrix} 3x + 5z & 3y + 5w \\ 2x + 4z & 2y + 4w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\leftrightsquigarrow \quad \begin{cases} 3x + 5z = 1 \\ 2x + 4z = 0 \end{cases}$ and $\begin{cases} 3y + 5w = 0 \\ 2y + 4w = 1 \end{cases}$
- $\leftrightsquigarrow \quad \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \rightsquigarrow$ ...row reduce... $\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5/2 \\ 0 & 1 & -1 & 3/2 \end{array}\right]$

# Warning (using the other condition)

- Consider again the matrix $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$.
- We need to find a matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AB = I_2 = BA$.
- Suppose we impose instead the condition $BA = I_2$.
- $BA = I_2 \quad \leftrightsquigarrow \quad \begin{bmatrix} 3x + 2y & 5x + 4y \\ 3z + 2w & 5z + 4w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\leftrightsquigarrow \quad \begin{cases} 3x + 2y = 1 \\ 5x + 4y = 0 \end{cases}$ and $\begin{cases} 3z + 2w = 0 \\ 5z + 4w = 1 \end{cases}$
- $\leftrightsquigarrow \quad \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \rightsquigarrow$ ...row reduce... $\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5/2 & 3/2 \end{array}\right]$
- **Morale:** We work with the transpose of $A$ and of $A^{-1}$.

## General Method for finding (if possible) the inverse

- Let $A$ be an $n \times n$ matrix. Finding a matrix $B$ with $AB = I_n$ results in $n$ linear systems, each consisting of $n$ equations in $n$ unknowns.
- The corresponding augmented matrices have the same matrix $A$ on their left side and a column of 0's and a single 1 on their right side.
- By solving these $n$ systems simultaneously, we can speed up the process of finding the inverse matrix.
- To do so, we construct the augmented matrix $[\, A \,|\, I_n \,]$. We row reduce to obtain, if possible, the augmented matrix $[\, I_n \,|\, B \,]$.

$$\left[\begin{array}{c|ccc} & 1 & & \\ A & & \ddots & \\ & & & 1 \end{array}\right] \quad \text{... row reduce ...} \quad \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & B \\ & & 1 & \end{array}\right]$$

- The matrix $B$, if it exists, is the inverse $A^{-1}$ of $A$.

## Example 2

Find the inverse of the $3 \times 3$ matrix $A = \begin{bmatrix} -1 & 3 & -1 \\ 2 & -2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

# General Formula for a 2x2 Matrix

- For simplicity we write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ instead of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.
- Construct the augmented matrix $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$.
- Perform the Gaussian Elimination Algorithm. Set $\Delta = ad - bc$.
- $\underset{\frac{1}{a}R_1}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \underset{R_2 - cR_1}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$
- $\underset{\frac{a}{\Delta}R_2}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{array}\right] \overset{R_1 - \frac{b}{a}R_2}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{\Delta} & -\frac{b}{\Delta} \\ 0 & 1 & -\frac{c}{\Delta} & \frac{a}{\Delta} \end{array}\right]$

## The Inverse of a $2 \times 2$ Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a $2 \times 2$ matrix.

- We define $\det(A) = ad - bc$.
- $A$ is invertible ($\equiv$ nonsingular) if and only if $\det(A) \neq 0$.

  In particular, $$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Looking back at the formula for $A^{-1}$, where A is a $2 \times 2$ matrix whose determinant is nonzero, we see that, to find the inverse of $A$

- we divide by the determinant of $A$,
- switch the diagonal elements of $A$,
- change the sign of the off-diagonal elements.

If the determinant is equal to 0, then the inverse of $A$ does not exist.

# Example 3

Find the inverse of the matrix

- $$A = \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$$

- $$B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The determinant can be defined for any $n \times n$ matrix. The general formula is computationally complicated for $n \geq 3$.

We mention the following important result. Part (2) below will be of particular interest to us in the near future.

**Theorem**

Suppose that $A$ is an $n \times n$ matrix, and $X$ and $\mathbf{0}$ are $n \times 1$ matrices. Then

- $A$ is **invertible** ($\equiv$ **nonsingular**) if and only if $\det(A) \neq 0$.
- The matrix equation ($\equiv$ system of linear equations) $AX = \mathbf{0}$ has a **nontrivial solution** $\iff$ $A$ is **singular** $\iff$ $\det(A) = 0$.

# Example 4

Find the solution of the following matrix equations ($\equiv$ systems of linear equations)

- $$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $$\begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$