MA 138 – Calculus 2 with Life Science Applications
Eigenvectors and Eigenvalues
(Section 9.3)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

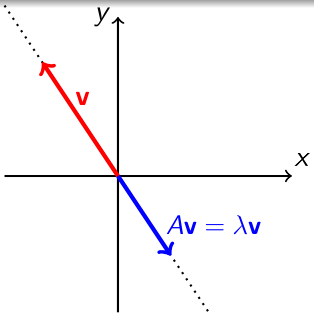
Eigenvalues and Eigenvectors

Definition

Assume that A is a square matrix. A nonzero vector $\mathbf{v}$ that satifies the equation

$$A\mathbf{v} = \lambda\mathbf{v} \qquad (\mathbf{v} \neq \mathbf{0})$$

is an **eigenvector** of the matrix A, and the number λ is an **eigenvalue** of the matrix A.



- The zero vector $\mathbf{0}$ always satisfies the equation $A\mathbf{0} = \lambda\mathbf{0}$ for any choice of λ. Thus $\mathbf{0}$ is not special. That's why we assume $\mathbf{v} \neq \mathbf{0}$.
- The eigenvalue λ can be 0, though.
- **Geometric interpretation**, when the eigenvalue $\lambda \in \mathbb{R}$: If we draw a straight line through the origin in the direction of an eigenvector, then any vector on this straight line will remain on the line after the map A is applied.

Example 1 (Online Homework # 5)

Determine if **v** is an eigenvector of the matrix A:

(a) $$A = \begin{bmatrix} -35 & -14 \\ 84 & 35 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix};$$

(b) $$A = \begin{bmatrix} 19 & 24 \\ -12 & -17 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix};$$

(c) $$A = \begin{bmatrix} -3 & -10 \\ 5 & 12 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}.$$

Example 2 (Online Homework # 6)

Given that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are eigenvectors of the matrix $A = \begin{bmatrix} 28 & 36 \\ -18 & -23 \end{bmatrix}$,

determine the corresponding eigenvalues λ_1 and λ_2.

Example 3 (Online Homework # 8)

Determine if λ is an eigenvalue of the matrix A:

(a) $$A = \begin{bmatrix} 7 & -10 \\ 0 & -3 \end{bmatrix} \quad \text{and} \quad \lambda = -2;$$

(b) $$A = \begin{bmatrix} -4 & -12 \\ 0 & 8 \end{bmatrix} \quad \text{and} \quad \lambda = 3;$$

(c) $$A = \begin{bmatrix} -92 & 42 \\ -196 & 90 \end{bmatrix} \quad \text{and} \quad \lambda = -8].$$

Finding the Eigenvalues

- We are interested in finding $\mathbf{v} \neq \mathbf{0}$ and λ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- We can rewrite this equation as $A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$.
- In order to factor $\mathbf{v}$, we must multiply $\lambda\mathbf{v}$ by the identity matrix I_2.

  (In the $n \times n$ case we multiply instead by I_n. The procedure and outcome are exactly the same!)
- Multiplication by I_2 yields $A\mathbf{v} - \lambda I_2\mathbf{v} = \mathbf{0}$.
- We can now factor $\mathbf{v}$, resulting in $(A - \lambda I_2)\mathbf{v} = \mathbf{0}$.
- In Section 9.2, we showed that in order to obtain a nontrivial solution ($\mathbf{v} \neq \mathbf{0}$), the matrix $A - \lambda I_2$ must be singular; that is,

$$\boxed{\det(A - \lambda I_2) = 0}$$

Let us make the previous calculations more explicit (in the 2×2 case):

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}} = \lambda \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}}$$

$$\Updownarrow$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}}_{A-\lambda I_2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation $\det(A - \lambda I_2) = 0$ that determines the eigenvalues of A is a polynomial equation in λ of degree two. This polynomial is referred to as the **characteristic polynomial** of A.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ define

- $\text{trace}(A) = a + d$,
- $\det(A) = ad - bc$.

The characteristic polynomial has a simple form:

Characteristic Polynomial of the 2×2 Matrix A

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

$$\Updownarrow$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\Updownarrow$$

$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$

Corollary

If λ_1 and λ_2 are the solutions of the characteristic polynomial, then they must satisfy

$$\text{trace}(A) = \lambda_1 + \lambda_2 \qquad \det(A) = \lambda_1 \lambda_2.$$

Example 4 (≈ Problems #49-56, Section 9.3, p. 534)

Consider the matrix $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$.

(a) Find its eigenvalues λ_1 and λ_2 .

(b) Find the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated with the eigenvalues from part (a).

(c) Graph the lines through the origin in the direction of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, together with the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and the vectors $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

Example 5 (Online Homework #10)

Find the eigenvalues and associated **unit** eigenvectors of the (symmetric) matrix $A = \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix}$.

Example 6 (Online Homework #12)

Let $A = \begin{bmatrix} -4 & 3 \\ 5 & k \end{bmatrix}$.

Find the value of k so that A has 0 as an eigenvalue.

Example 7 (Online Homework #13)

For which value of k does the matrix $A = \begin{bmatrix} -3 & k \\ -8 & -8 \end{bmatrix}$ have one real eigenvalue of multiplicity 2?

Example 8 (Online Homework #16)

Find a matrix A such that $\mathbf{v}_1 = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ are eigenvectors of A, with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ respectively.

Example 9 (Complex Eigenvalues)

Consider the matrix $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$.

(a) Find its eigenvalues.

(b) Find the eigenvectors associated with the eigenvalues from part (a).