MA 138 – Calculus 2 with Life Science Applications
Tangent Planes, Differentiability, and Linearization
(Section 10.4)

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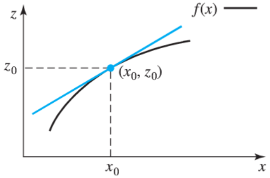
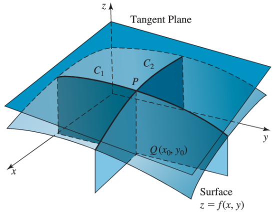
- The key idea in both the one- and the two-dimensional case is to approximate functions by linear functions, so that the error in the approximation vanishes as we approach the point at which we approximated the function.

- If $z = f(x)$ is differentiable at $x = x_0$, then the equation of the tangent line of $z = f(x)$ at (x_0, z_0) with $z_0 = f(x_0)$ is given by

$$z - z_0 = f'(x_0)(x - x_0).$$



- We now generalize this situation to functions of two variables. The analogue of a tangent line is called a **tangent plane**, an example of which is shown in the picture on the right.



Tangent Plane

- Let $z = f(x, y)$ be a function of two variables.
- We saw that the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$, evaluated at (x_0, y_0), are the slopes of tangent lines at the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$, to certain curves through (x_0, y_0, z_0) on the surface $z = f(x, y)$.
- These two tangent lines, one in the x-direction, the other in the y-direction, define a unique plane.
- If, in addition, $f(x, y)$ has partial derivatives that are continuous on an open disk containing (x_0, y_0), then we can show that the tangent line at (x_0, y_0, z_0) to any other smooth curve on the surface $z = f(x, y)$ through (x_0, y_0, z_0) is contained in this plane.
- The plane is then called the tangent plane.

More precisely, one can show the following result:

Equation of the Tangent Plane

If the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, **exists**, then that tangent plane has the equation

$$z - z_0 = \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0).$$

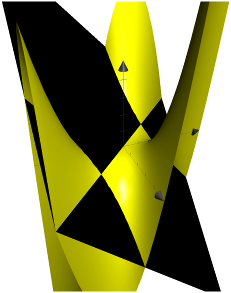
- We should observe the similarity of this equation to the equation of the tangent line in the one-dimensional case.
- As we mentioned, the mere existence of the partial derivatives $\dfrac{\partial f(x_0, y_0)}{\partial x}$ and $\dfrac{\partial f(x_0, y_0)}{\partial y}$ is not enough to guarantee the existence of a tangent plane at (x_0, y_0, z_0); something stronger is needed.

Example 1

Find an equation of the tangent plane to surface given by the graph of the function

$$z = f(x, y) = xy^2 + x^2y$$

at the point $(1, -1, 0)$.



Example 2 (Problem #4, Online Homework)

Find an equation of the tangent plane to surface given by the graph of the function

$$F(r, s) = r^4 s^{-0.5} - s^{-4}$$

at the point with $r_0 = 1$ and $s_0 = 1$.

Review of differentiability for a function of one variable

If $z = f(x)$ is a function of one variable, the tangent line is used to approximate $f(x)$ at $x = x_0$. The linearization of $f(x)$ at $x = x_0$ is given by

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

The distance between $f(x)$ and its linear approximation at $x = x_0$ is then

$$|f(x) - L(x)| = |f(x) - f(x_0) - f'(x_0)(x - x_0)|.$$

If we divide the latter equation by the distance $|x - x_0|$, we find that

$$\left|\frac{f(x) - L(x)}{x - x_0}\right| = \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right|.$$

Taking a limit and using the definition of the derivative at $x = x_0$, yields

$$\lim_{x \to x_0} \left|\frac{f(x) - L(x)}{x - x_0}\right| = 0.$$

We say that $f(x)$ is differentiable at $x = x_0$ if the above equation holds.

Differentiability and Linearization

Suppose that $f(x, y)$ is a function of two independent variables with both $\partial f/\partial x$ and $\partial f/\partial y$ defined on an open disk containing (x_0, y_0).

- Set $L(x, y) = f(x_0, y_0) + \dfrac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \dfrac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$.
- $f(x, y)$ is differentiable at (x_0, y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \left| \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right| = 0$.
- If $f(x, y)$ is differentiable at (x_0, y_0), then $z = L(x, y)$ provides the equation of the tangent plane to the graph of f at (x_0, y_0, z_0).
- $f(x, y)$ is differentiable if it is differentiable at every point of its domain.
- Suppose f is differentiable at (x_0, y_0), the approximation $f(x, y) \approx L(x, y)$ is the **standard linear approximation**, or the tangent plane approximation, of $f(x, y)$ at (x_0, y_0).

- That $f(x, y)$ is differentiable at (x_0, y_0) means that the function $f(x, y)$ is close to the tangent plane at (x_0, y_0) for all (x, y) close to (x_0, y_0).
- As in the one-dimensional case, the following theorem holds:

Theorem

If $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

- The mere existence of the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ at (x_0, y_0), however, is not enough to guarantee differentiability (and, consequently, the existence of a tangent plane at a certain point).
- The following differentiability criterion suffices for all practical purposes.

Sufficient Condition For Differentiability

Suppose $f(x, y)$ is defined on an open disk centered at (x_0, y_0) and the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are continuous on an open disk centered at (x_0, y_0). Then $f(x, y)$ is differentiable at (x_0, y_0).

Example 3 (Problem #6, Online Homework)

Estimate $f(3.01, 2.02)$ given that

$$f(3,2) = 4 \qquad f_x(3,2) = -5 \qquad f_y(3,2) = 2.$$

Example 4 (Problem #5(b), Exam 4, Spring 2012)

Find the linear approximation of the function

$$f(x, y) = x \cdot e^{xy}$$

at $(1, 0)$, and use it to approximate $f(1.1, -0.1)$. Using a calculator, compare the approximation with the exact value of $f(1.1, -0.1)$.

Example 5 (Problem #9, Online Homework)

Find the linearization of the function

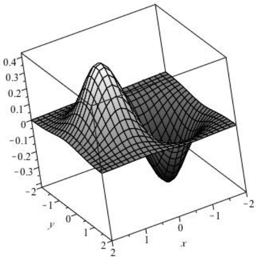
$$f(x, y) = \sqrt{23 - x^2 - 5y^2}$$

at the point $(-3, -1)$.

Use the linear approximation to estimate the value of $f(-3.1, -0.9)$.

Example 6 (Problem #6, Exam 3, Spring 2013)

Consider the function $f(x, y) = x\,e^{-x^2-y^2}$ whose graph is given in the picture on the right.



(*a*) Find the z-coordinate z_0 of the point P on the graph of the function $f(x, y)$ with x-coordinate $x_0 = 1$ and y-coordinate $y_0 = 1$.

(*b*) Write the equation of the tangent plane to the graph of the function $f(x, y)$ at the point P, as above, with coordinates $x_0 = 1$ and $y_0 = 1$.

(*c*) Write the linear approximation, $L(x, y)$, of the function f at the point with $x_0 = 1$ and $y_0 = 1$, as above, and use it to approximate $f(1.1, 0.9)$.

Compare this approximate value to the exact value $f(1.1, 09)$.

Functions of more than two variables

Similar discussions can be carried for functions of more than two variables.

For example, if $w = f(x, y, z)$ is a function of three independent variables which is differentiable at a point (x_0, y_0, z_0), then the linear approximation $L(x, y, z)$ of f at (x_0, y_0, z_0) is given by the formula

$$L(x, y, z) =$$

$$f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0) \cdot (x - x_0) + f_y(x_0, y_0, z_0) \cdot (y - y_0) + f_z(x_0, y_0, z_0) \cdot (z - z_0).$$

Example 7 (Problem #10, Online Homework)

Find the linear approximation to the function

$$f(x, y, z) = \frac{xy}{z}$$

at the point $(-2, -3, -1)$.