MA 138 – Calculus 2 with Life Science Applications
Linear Systems: Theory
(Section 11.1)

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Example 2 (Problem #9, Exam 3, Spring 2013)

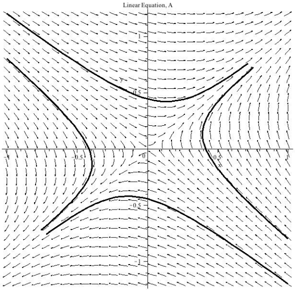
Let $A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$.

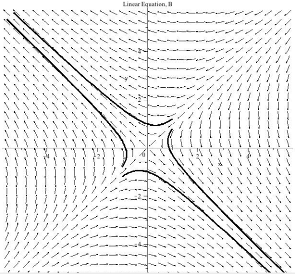
(*a*) Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors of A. What are the corresponding eigenvalues?

(*b*) Find the general solution of the system of differential equations

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

(c) Which direction field corresponds to the system of DEs in (b)?





- Sketch the lines in the direction of the eigenvectors.
- Indicate on each line the direction in which the solution would move if the initial condition is on that line.
- From your analysis, the point $(0, 0)$ is a: (choose one)

  □ sink (stable equilibrium)

  □ saddle point (unstable equilibrium)

  □ source (unstable equilibrium)

Example 3 (Problem # 8, Exam 4, Spring 2014)

(Metapopulations). Many biological populations are subdivided into smaller subpopulations with limited movement between them. The entire collection of such subpopulations is called a **metapopulation**. Consider the following model of two subpopulations, where x_1 and x_2 are the number of individuals in each:

$$\frac{dx_1}{dt} = r_1 x_1 - m_1 x_1 + m_2 x_2 \qquad \frac{dx_2}{dt} = r_2 x_2 - m_2 x_2 + m_1 x_1.$$

Here r_i is the intrinsic growth rate of subpopulation i and m_i is the per capita movement rate from patch i into the other patch.

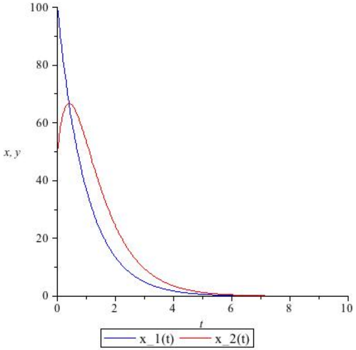
(a) Suppose $r_1 = 1, \quad r_2 = -2, \quad m_1 = 2, \quad$ and $\quad m_2 = 0.$ Write the system of differential equations corresponding to these choices.

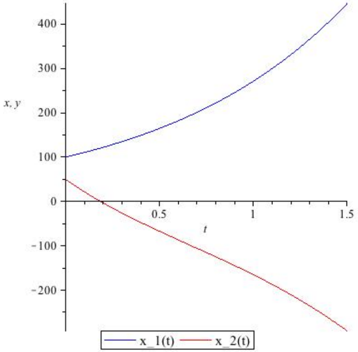
(b) Find the general solution to the system in (a).

Example 3 (cont'd)

(*c*) Find the solution to the system in (*a*) when the initial size of each subpopulation is $x_1(0) = 100$ and $x_2(0) = 50$. What happens to the two subpopulations as $t \to \infty$?

(*d*) Which of the following plots describes what happens to the two subpopulations?





Example 4 (Problem # 9, Exam 4, Spring 2013)

If a large block of ice is placed in a room we can describe how the temperature of the block of ice and room are changing using the system of differential equations

$$\frac{dI}{dt} = \alpha(R - I) \qquad \frac{dR}{dt} = \beta(I - R),$$

where I is the temperature of the block of ice, R is the temperature of the room and α and β are positive constants that determine the relationship between the rates of change of the temperatures of the block of ice and room and the difference between these temperatures. (All temperatures are measured in degrees Fahrenheit (°F).)

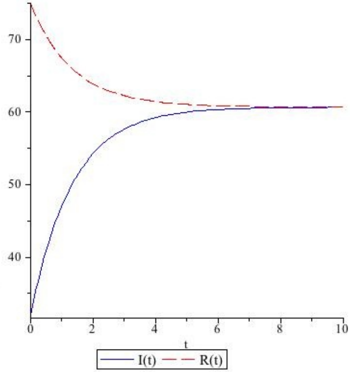
(a) Find the solution of the given system of linear differential equations in the case that $\alpha = 0.5$, $\beta = 0.25$, $I(0) = 32$, and $R(0) = 74$. That is

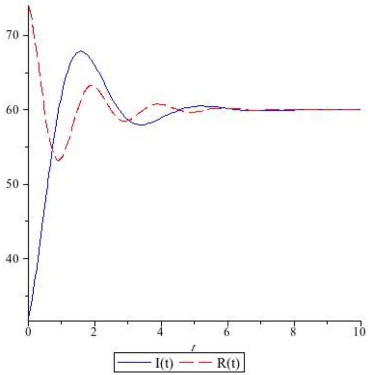
$$\frac{d}{dt}\begin{bmatrix} I \\ R \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ 0.25 & -0.25 \end{bmatrix}\begin{bmatrix} I \\ R \end{bmatrix} \qquad \begin{bmatrix} I(0) \\ R(0) \end{bmatrix} = \begin{bmatrix} 32 \\ 74 \end{bmatrix}$$

Example 4 (cont'd)

(*b*) Describe the long term behavior of your solution. In particular what happens to the temperature of the room and to the block of ice?

(*c*) Which of the pictures below describes the behavior of the two temperatures?





Example 5 (Problem # 6, Exam 4, Spring 2012)

Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. It is easy to verify that A has repeated eigenvalues $\lambda_1 = \lambda_2 = 1$ and that every eigenvector of A is of the form $c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where c is a real number different from 0.

Consider, now, the corresponding system of linear differential equations given by the above matrix:

$$\begin{cases} \dfrac{dx_1}{dt} & = & x_1 + 2x_2 \\ \dfrac{dx_2}{dt} & = & x_2 \end{cases}$$

(or, more concisely, $\dfrac{d\mathbf{x}}{dt} = A \cdot \mathbf{x}(t)$.)

Example 5 (cont'd)

(*a*) Show that $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a solution of the given system of differential equations.

(*b*) Show that $\mathbf{x}(t) = te^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$ is a (second independent) solution of the given system of differential equations.

The case of complex eigenvalues and eigenvectors

Suppose that the linear system of DEs $d\mathbf{x}/dt = A\mathbf{x}$, where A is a 2×2 matrix with real coefficients, has two complex (conjugate) eigenvalues $\lambda_{1,2} = a \pm ib$ with corresponding eigenvectors $\mathbf{v}_{1,2} = \mathbf{u} \pm i\mathbf{w}$, where $\mathbf{u}$ and $\mathbf{w}$ are real-valued vectors.

Using Euler's formula $e^{x+iy} = e^x(\cos y + i \sin y)$, it can be shown that the typical solutions of the DEs can be written as

$$e^{(a \pm ib)t}(\mathbf{u} \pm i\mathbf{w}) = \mathbf{g}(t) \pm i\mathbf{h}(t)$$

where both $\mathbf{h}(t)$ and $\mathbf{g}(t)$ form the family of real solutions of the given system of DEs.

More precisely

$$\mathbf{g}(t) = e^{at}(\mathbf{u}\cos(bt) - \mathbf{w}\sin(bt)) \qquad \mathbf{h}(t) = e^{at}(\mathbf{w}\cos(bt) + \mathbf{u}\sin(bt)).$$

Example 6 (Online Homework #9)

Consider the linear system

$$\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \mathbf{y}.$$

(*a*) Find the eigenvalues and eigenvectors for the coefficient matrix. Write the system of differential equations corresponding to these choices.

(*b*) Find the real valued solution to the initial value problem

$$\begin{cases} y_1' = 3y_1 + 2y_2 & y_1(0) = -9, \\ y_2' = -5y_1 - 3y_2 & y_2(0) = 10. \end{cases}$$

Use t as the independent variable in your answer.