MA 138 – Calculus 2 with Life Science Applications
Linear Systems: Theory
(Section 11.1)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Equilibria and Stability

- In Section 8.2 we already encountered the concepts of equilibria and stability, when we discussed ordinary DEs. Both concepts can be extended to systems of DEs.
- We now restrict ourselves to the case

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad \text{with} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

- We say that a point $\widehat{\mathbf{x}} = \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix}$ is an **equilibrium point** of our given system of linear DEs whenever $\qquad A\widehat{\mathbf{x}} = \mathbf{0}$.
- It follows from results in Section 9.2 that if $\det A \neq 0$, then $(0, 0)$ is the only equilibrium of our given system of linear DEs. If $\det A = 0$, then there will be other equilibria.

- If we start a system of DEs at an equilibrium, it remains there at all later times.
- This does not mean that if the system is in equilibrium and is perturbed by a small amount, it will return to the equilibrium.
- Whether or not a solution will return to an equilibrium after a small perturbation is addressed by the **stability** of the equilibrium.
- In the case when the matrix A has two real and distinct eigenvalues, the solution of our given system of linear DEs is given by

  $$x(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2,$$

  where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors corresponding to the eigenvalues λ_1 and λ_2 of A and the constants c_1 and c_2 depend on the initial condition.
- Knowing the solution will allow us to study the behavior of the solutions as $t \to \infty$ and thus address the question of stability, at least when the eigenvalues are distinct.

Classification of Equilibria

Case 1: A has **two distinct real nonzero eigenvalues** λ_1 and λ_2

1. **Both eigenvalues are negative:** The equilibrium $(0, 0)$ is **globally stable**, since the solution will approach the equilibrium $(0, 0)$ regardless of the starting point. We call $(0, 0)$ a **sink** or a **stable node**.

2. **The eigenvalues have opposite signs:** Unless we start in the direction of the eigenvector associated with the negative eigenvalue, the solution will not converge to the equilibrium $(0, 0)$. We say that the equilibrium $(0, 0)$ is **unstable** and call $(0, 0)$ a **saddle point**.

3. **Both eigenvalues are positive:** The solution will not converge to $(0, 0)$ unless we start at $(0, 0)$. We say that the equilibrium $(0, 0)$ is **unstable**, and we call $(0, 0)$ a **source** or an **unstable node**.

Case 2: A has **two complex conjugate eigenvalues** $\lambda_{1,2} = \alpha \pm i\beta$

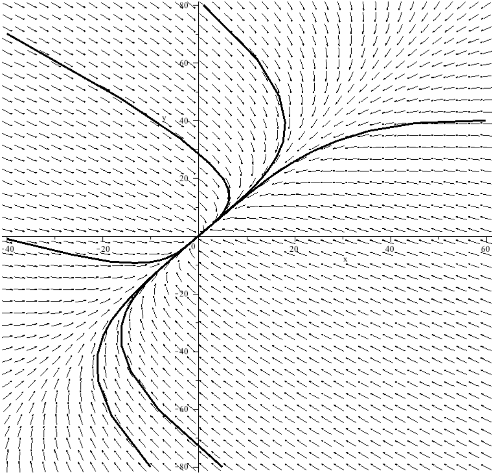
1. $\alpha < 0$: Starting from any point other than $(0,0)$, solutions spiral into the equilibrium $(0,0)$. For this reason, the equilibrium $(0,0)$ is called a **stable spiral**. When we plot solutions as functions of time, they show oscillations. The amplitude of the oscillations decreases over time. We therefore call the oscillations **damped**.
2. $\alpha > 0$: Starting from any point other than $(0,0)$, the solutions spiral out from the equilibrium $(0,0)$. For this reason, we call the equilibrium $(0,0)$ an **unstable spiral**. When we plot solutions as functions of time, we see that the solutions show oscillations as before, but this time their amplitudes are increasing.
3. $\alpha = 0$: Solutions spiral around the equilibrium $(0,0)$, but neither approach nor move away from the equilibrium (since the amplitude of the solutions does not change). The equilibrium $(0,0)$ is called a **neutral spiral** or a **center**. The solutions form closed curves.

Example 8

(0, 0) globally stable equilibrium; sink or a stable node

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

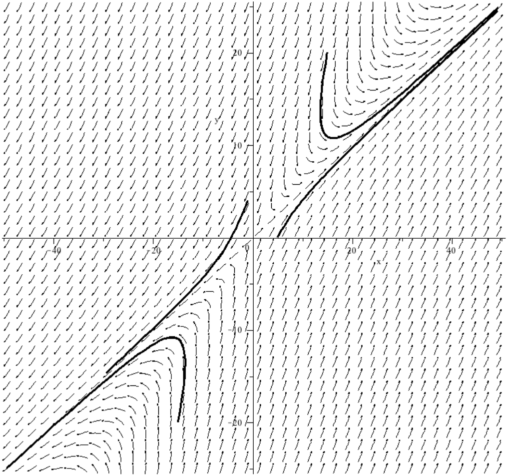
$$\lambda_1 = -1 \qquad \lambda_2 = -4$$



Example 9 (0, 0) unstable equilibrium; saddle point

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

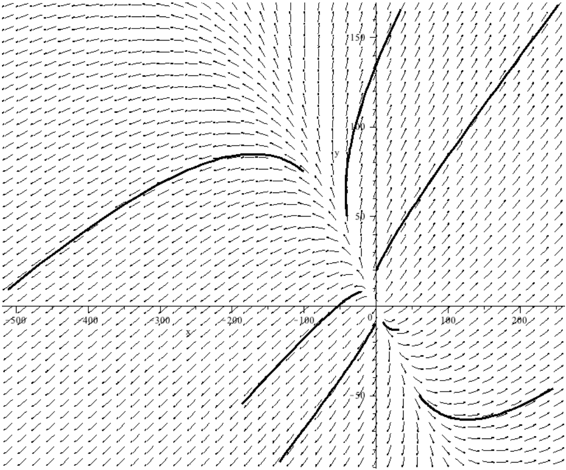
$$\lambda_1 = 1 \qquad \lambda_2 = -2$$



Example 10 $(0,0)$ unstable node; source

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

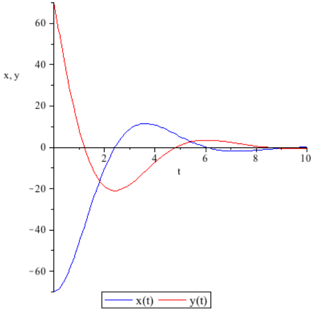
$\lambda_1 = 1 \qquad \lambda_2 = 4$

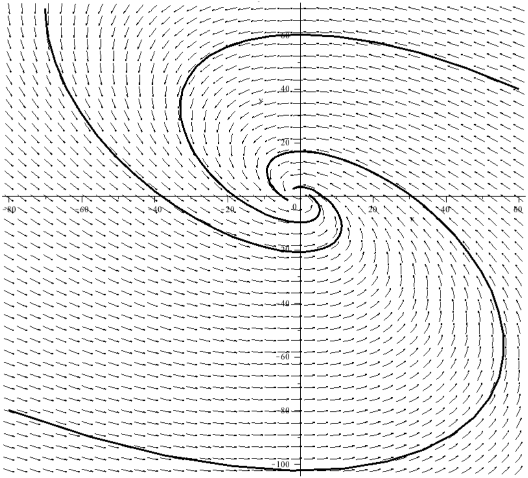


Example 11 (0, 0) stable spiral; negative real part

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

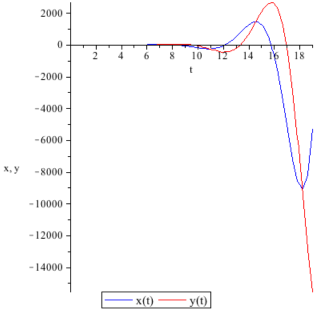


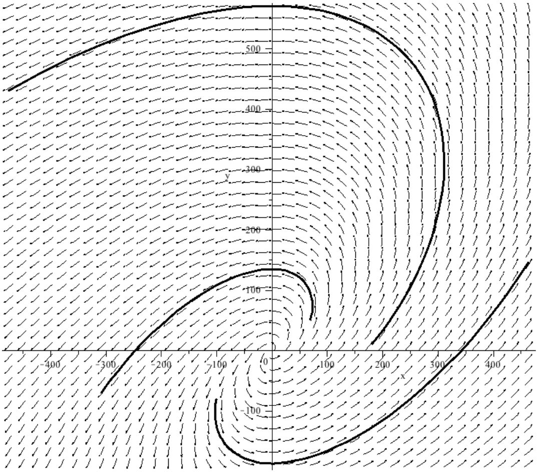


Example 12 (0, 0) unstable spiral; positive real part

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

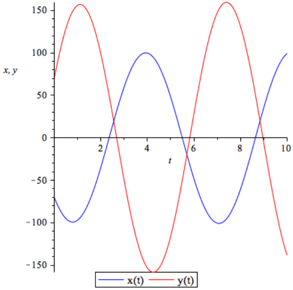


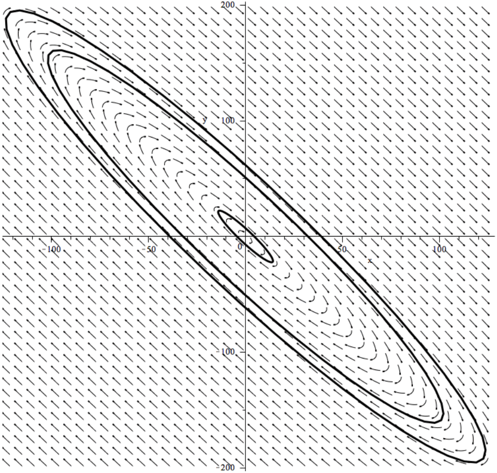


Example 13 (0, 0) neutral spiral or center; no real part

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_{1,2} = \pm i$$





Summary of Stability at $(0,0)$

- The relationships between the characteristic polynomial, eigenvalues, trace, and determinant of a 2×2 matrix A are given by

$$\det(A - \lambda I_2) = \lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$$

$$\text{trace}(A) = \text{Re}(\lambda_1) + \text{Re}(\lambda_2) \qquad \det(A) = \lambda_1 \lambda_2.$$

- Also, the discriminant of the quadratic equation $\det(A - \lambda I_2) = 0$ is

$$\Delta = [\text{trace}(A)]^2 - 4\det(A)$$

Thus the condition $\Delta = 0$ ($\equiv$repeated eigenvalue) describes the parabola $\det(A) = 1/4[\text{trace}(A)]^2$ in the trace-det plane.

- **Theorem:** The origin $(0,0)$ of a system of two linear, homogeneous DEs with constant coefficients is a stable equilibrium $\Leftrightarrow$ the real parts of both eigenvalues are negative $\Leftrightarrow$ $\det(A) > 0$ and $\text{trace}(A) < 0$.

The stability properties of the equilibrium at the origin can be summarized graphically in terms of the determinant and the trace of the matrix A in the trace-det plane: