MA 138 – Calculus 2 with Life Science Applications
Linear Systems: Applications
(Section 11.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

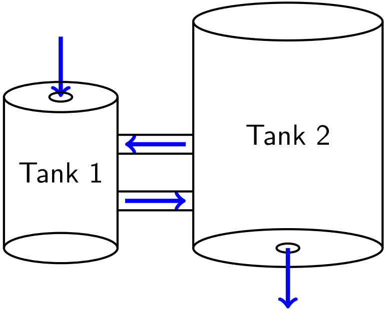
Department of Mathematics
University of Kentucky

Compartment Models

- Compartment models describe flow between compartments, such as nutrient flow between lakes or the flow of a radioactive tracer between different parts of an organism.
- In the simplest situations, the resulting model is a system of linear differential equations.

Example 1 (Online Homework #4)

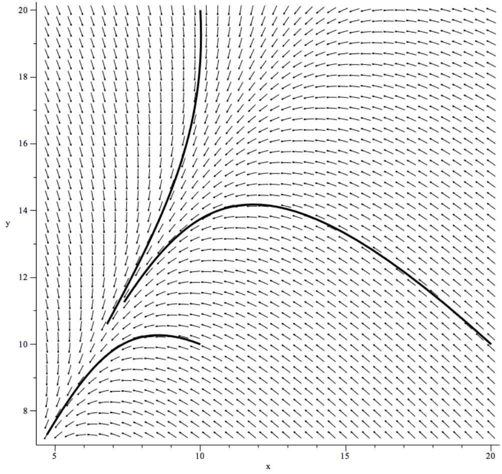
Consider two brine tanks connected as shown in the figure. Pure water flows into the top of Tank 1 at a rate of 15 L/min. The brine solution is pumped from Tank 1 into Tank 2 at a rate of 40 L/min, and from Tank 2 into Tank 1 at a rate of 25 L/min. A brine solution flows out the bottom of Tank 2 at a rate of 15 L/min.

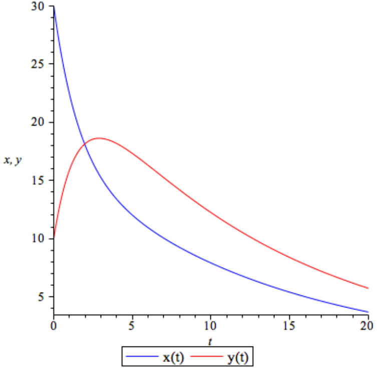


Suppose there are 100 L of brine in Tank 1 and 120 L of brine in Tank 2. Let x be the amount of salt, in kilograms, in Tank 1 after t minutes, and y the amount of salt, in kilograms, in Tank 2 after t minutes.

Assume that each tank is mixed perfectly. Set up a system of first-order differential equations that models this situation.

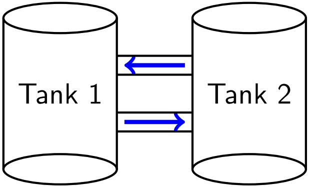
Example 1: The direction field and the graph of two particular solutions of the system of linear DEs are plotted below:





Example 2 (Online Homework #5)

Consider two brine tanks connected as shown in the figure. The brine solution is pumped from Tank 1 into Tank 2 at a rate of 10 L/min, and from Tank 2 into Tank 1 at a rate of 10 L/min. Suppose there are 50 L of brine in Tank 1 and 25 L of brine in Tank 2.



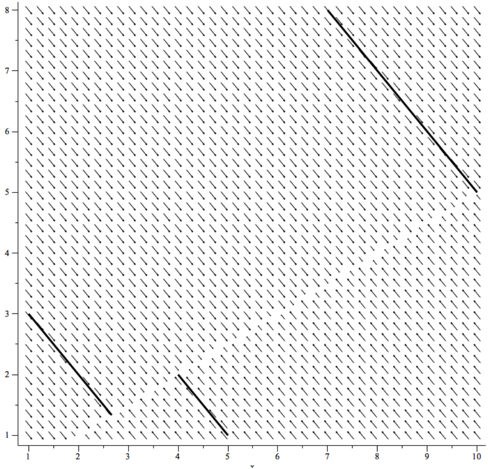
Let x be the amount of salt, in kilograms, in Tank 1 after t minutes have elapsed, and let y the amount of salt, in kilograms, in Tank 2 after t minutes have elapsed.

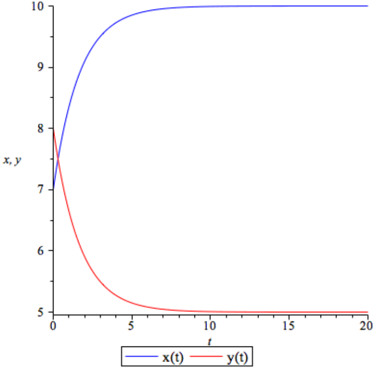
Assume that each tank is mixed perfectly.

If $x(0) = 7$ kg and $y(0) = 8$ kg, find the amount of salt in each tank after t minutes.

As $t \to \infty$, how much salt is in each tank?

Example 2: The direction field and the graph of the two solutions of the system of linear DEs with given initial conditions are plotted below:





Higher Order Differential Equations

- (Ordinary) differential equations ($\equiv$ODEs) arise naturally in many different contexts throughout mathematics and science (social and natural). Indeed, the most accurate way of describing changes mathematically uses differentials and derivatives.
- So far we have looked only to first order differential equations.
- A simple example is Newton's Second Law of Motion, which is described by the differential equation $\quad m\dfrac{d^2x(t)}{dt^2} = F(x(t))$ (m is the constant mass of a particle subject to a force F, which depends on the position $x(t)$ of the particle at time t).
- Let F be a given function of x, y, and derivatives of y. Then an equation of the form

$$y^{(n)} = F\left(x, y, y', \cdots y^{(n-1)}\right)$$

is called an explicit ordinary differential equation of order n.

Reduction of to a First-Order System

- Differential equations can usually be solved more easily if the order of the equation can be reduced.
- Any differential equation of order n,

$$y^{(n)} = F\left(x, y, y', y'', \ \cdots, \ y^{(n-1)}\right)$$

can be written as a system of n first-order differential equations by defining a new family of unknown functions

$$y_i = y^{(i-1)}$$

for $i = 1, 2, \ldots, n$.
- Note that these new functions are related by

$$y'_1 = y_2 \qquad y'_2 = y_3 \quad \cdots \quad y'_{n-1} = y_n \qquad y'_n = F(x, y_1, y_2, \ldots, y_n).$$

- Your solution is then the function $y_1 = y$.

Example 3 (Online Homework #2)

Solve the following differential equation:

$$y'' - 3y' - 10y = 0$$

with the initial conditions $y = 1$, $y' = 10$ at $x = 0$.