11.3 Nonlinear Autonomous Systems: Theory

In this section, we will develop some of the theory needed to analyze systems of differential equations of the form

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{11.47}$$

where $f_i : \mathbf{R}^n \to \mathbf{R}$, for $i = 1, 2, \ldots, n$. We assume that the functions $f_i, i = 1, 2, \ldots, n$, do not explicitly depend on t; the system (11.47) is therefore called *autonomous*. We no longer assume that the functions f_i are linear, as in Section 11.1. Using vector notation, we can write the system (11.47) in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]'$, and $\mathbf{f}(\mathbf{x})$ is a vector-valued function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ with components $f_i(x_1, x_2, \ldots, x_n) : \mathbf{R}^n \to \mathbf{R}$, $i = 1, 2, \ldots, n$. The function $\mathbf{f}(\mathbf{x})$ defines a direction field, just as in the linear case.

Unless the functions f_i are linear, it is typically not possible to find explicit solutions of systems of differential equations. If we want to solve such systems, we frequently must use numerical methods. Instead of trying to find solutions, we will focus on point equilibria and their stability, just as in Section 8.2.

The definition of a point equilibrium (as given in Section 8.2) must be extended to systems of the form (11.47). We say that a point

$$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$$

is a point equilibrium (or simply equilibrium) of (11.47) if

$$\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{0}$$

An equilibrium is also called a **critical point**. As in the linear case, this is a point in the direction field at which the direction vector has length 0, implying that if we start a system of differential equations at an equilibrium point, it will stay there for all later times.

As in the linear case, a solution might not return to an equilibrium after a small perturbation; this possibility is addressed by the stability of the equilibrium. The theory of stability for systems of nonlinear autonomous differential equations is parallel to that in Section 8.2; there is both an analytical and graphical approach that reduces to the theory set forth there when there is a single differential equation. We will restrict our discussion to systems of two equations in two variables. (The concepts are the same when we have more than two equations, but the calculations become more involved.)

11.3.1 Analytical Approach

A Single Autonomous Differential Equation

EXAMPLE 1 Find all equilibria of

$$\frac{dx}{dt} = x(1 - x) \tag{11.48}$$

and analyze their stability.

Solution We developed the theory for single autonomous differential equations in Section 8.2. To find equilibria, we set

$$x(1-x) = 0$$

which yields

$$\hat{x}_1 = 0 \quad \text{and} \quad \hat{x}_2 = 1$$

To analyze the stability of these equilibria, we linearize the differential equation (11.48) about each equilibrium and compute the corresponding eigenvalue. We set

$$f(x) = x(1-x)$$

Then

$$f'(x) = 1 - 2x$$

The eigenvalue associated with the equilibrium $\hat{x}_1 = 0$ is

$$\lambda_1 = f'(0) = 1 > 0$$

which implies that $\hat{x}_1 = 0$ is unstable.

The eigenvalue associated with the equilibrium $\hat{x}_2 = 1$ is

$$\lambda_2 = f'(1) = -1 < 0$$

which implies that $\hat{x}_2 = 1$ is locally stable. ■

The eigenvalue corresponding to an equilibrium of the differential equation

$$\frac{dx}{dt} = f(x) \tag{11.49}$$

is the slope of the function $f(x)$ at the equilibrium value. The reason for this is discussed in detail in Section 8.2; we repeat the basic argument here. Suppose that $\hat{x}$ is an equilibrium of (11.49); that is, $f(\hat{x}) = 0$. If we perturb $\hat{x}$ slightly (i.e., if we look at $\hat{x} + z$ for small $|z|$), we can find out what happens to $\hat{x} + z$ by examining

$$\frac{dx}{dt} = \frac{d}{dt}(\hat{x} + z) = \frac{dz}{dt}$$

Since the perturbation is small, we can linearize

$$f(\hat{x} + z) \approx f(\hat{x}) + f'(\hat{x})(\hat{x} + z - \hat{x}) = f'(\hat{x})z$$

[In the last step, we used the fact that $f(\hat{x}) = 0$.] We find that

$$\frac{dz}{dt} \approx f'(\hat{x})z$$

which has the approximate solution

$$z(t) \approx z(0)e^{\lambda t} \quad \text{with } \lambda = f'(\hat{x})$$

Therefore, if $f'(\hat{x}) < 0$, then $z(t) \to 0$ as $t \to \infty$ and, hence, $x(t) = \hat{x} + z(t) \to \hat{x}$ as $t \to \infty$; that is, the solution will return to the equilibrium $\hat{x}$ after a small perturbation. In this case, $\hat{x}$ is locally stable. If $f'(\hat{x}) > 0$, then $z(t)$ will not go to 0, which implies that $\hat{x}$ is unstable. The linearization of $f(x)$ thus tells us whether an equilibrium is locally stable or unstable. We will use linearization as well to determine the stability of equilibria of systems of differential equations.

Systems of Two Differential Equations We consider differential equations of the form

$$\frac{dx_1}{dt} = f_1(x_1, x_2) \tag{11.50}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2) \tag{11.51}$$

or, in vector notation,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \tag{11.52}$$

where $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))$ and $\mathbf{f}(\mathbf{x})$ is a vector-valued function $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$ with $f_i(\mathbf{x}) : \mathbf{R}^2 \to \mathbf{R}$. An equilibrium or critical point, $\hat{\mathbf{x}}$, of (11.52) satisfies

$$\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{0}$$

Suppose that $\hat{\mathbf{x}}$ is a point equilibrium. Then, as in the case of one nonlinear equation, we look at what happens to a small perturbation of $\hat{\mathbf{x}}$. We perturb $\hat{\mathbf{x}}$; that is, we look at how $\hat{\mathbf{x}} + \mathbf{z}$ changes under the dynamics described by (11.52):

$$\frac{d}{dt}(\hat{\mathbf{x}} + \mathbf{z}) = \frac{d}{dt}\mathbf{z} = \mathbf{f}(\hat{\mathbf{x}} + \mathbf{z})$$

The linearization of $\mathbf{f}(\mathbf{x})$ about $\mathbf{x} = \hat{\mathbf{x}}$ is

$$\mathbf{f}(\hat{\mathbf{x}}) + D\mathbf{f}(\hat{\mathbf{x}})\mathbf{z} = D\mathbf{f}(\hat{\mathbf{x}})\mathbf{z}$$

where we used the fact that $\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{0}$. The matrix $D\mathbf{f}(\hat{\mathbf{x}})$ is the Jacobi matrix evaluated at $\hat{\mathbf{x}}$. If we approximate $\mathbf{f}(\hat{\mathbf{x}} + \mathbf{z})$ by its linearization $D\mathbf{f}(\hat{\mathbf{x}})$, then

$$\frac{d\mathbf{z}}{dt} = D\mathbf{f}(\hat{\mathbf{x}})\mathbf{z} \tag{11.53}$$

is the linear approximation of the dynamics of the perturbation $\mathbf{z}$.

We now have a system of linear differential equations that is a good approximation, provided that $\mathbf{z}$ is sufficiently close to $\mathbf{0}$. In Section 11.1, we learned how to analyze linear systems. We saw that eigenvalues of the matrix $D\mathbf{f}(\hat{\mathbf{x}})$ allowed us to determine the nature of the equilibrium. We will use the same approach here, but we emphasize that this is now a *local* analysis, just as in the case of a single differential equation, since we know that the linearization (11.53) is a good approximation only as long as we are sufficiently close to the point about which we linearized.

We return to our classification scheme for the linear case,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))$ and A is a 2×2 matrix. We let

$$\Delta = \det A \qquad \text{and} \qquad \tau = \operatorname{tr} A$$

When we linearize a nonlinear system about an equilibrium, the matrix A is the Jacobi matrix evaluated at the equilibrium:

$$A = D\mathbf{f}(\hat{\mathbf{x}})$$

We exclude the following cases: (i) $\Delta = 0$ (when $\Delta = 0$, at least one eigenvalue is equal to 0), (ii) $\tau = 0$ and $\Delta > 0$ (when $\tau = 0$ and $\Delta > 0$, the two eigenvalues are purely imaginary), and (iii) $\tau^2 = 4\Delta$ (when $\tau^2 = 4\Delta$, the two eigenvalues are identical). Except in these three cases, we can use the same classification scheme as in the linear case (Figure 11.35).

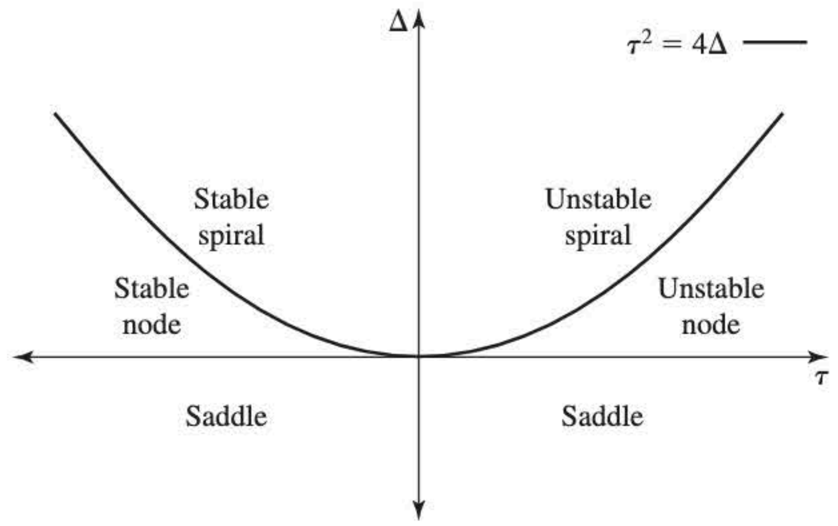

Figure 11.35 The stability behavior of a system of two autonomous equations.

The extension from the linear case is possible because the linearized vector field and the original vector field are geometrically similar close to an equilibrium point. (After all, that is the idea behind linearization.) This result is known as the *Hartman–Grobman theorem*, which says that as long as $D\mathbf{f}(\hat{\mathbf{x}})$ has no zero or purely imaginary eigenvalues, then the linearized and the original vector fields are similar close to the equilibrium. That is,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \qquad \text{and} \qquad \frac{d\mathbf{z}}{dt} = D\mathbf{f}(\hat{\mathbf{x}})\mathbf{z}$$

behave similarly for $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$ with $\mathbf{z}$ close to $\mathbf{0}$.

We find the same classification scheme as in the linear case:

- The equilibrium $\hat{\mathbf{x}}$ is a *node* if both eigenvalues of $D\mathbf{f}(\hat{\mathbf{x}})$ are real, distinct, nonzero, and of the same sign. The node is locally stable if the eigenvalues are negative and unstable if the eigenvalues are positive.
- The equilibrium $\hat{\mathbf{x}}$ is a saddle point if both eigenvalues are real and nonzero but have opposite signs. A saddle point is unstable.
- The equilibrium $\hat{\mathbf{x}}$ is a spiral if both eigenvalues are complex conjugates with nonzero real parts. The spiral is locally stable if the real parts of the eigenvalues are negative and unstable if the real parts of the eigenvalues are positive.

When the two eigenvalues are purely imaginary, we cannot determine the stability by linearization.

EXAMPLE 2 Consider

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - 2x_1^2 - 2x_1x_2 \\ \frac{dx_2}{dt} &= 4x_2 - 5x_2^2 - 7x_1x_2 \end{aligned} \tag{11.54}$$

(a) Find all equilibria of (11.54) and **(b)** analyze their stability.

Solution **(a)** To find equilibria, we set the right-hand side of (11.54) equal to 0:

$$x_1 - 2x_1^2 - 2x_1x_2 = 0 \tag{11.55}$$

$$4x_2 - 5x_2^2 - 7x_1x_2 = 0 \tag{11.56}$$

Factoring out x_1 in the first equation and x_2 in the second yields

$$x_1(1 - 2x_1 - 2x_2) = 0 \qquad \text{and} \qquad x_2(4 - 5x_2 - 7x_1) = 0$$

That is,

$$x_1 = 0 \qquad \text{or} \qquad 2x_1 + 2x_2 = 1$$

and

$$x_2 = 0 \qquad \text{or} \qquad 7x_1 + 5x_2 = 4$$

Combining the different solutions, we get the following four cases:

(i) $x_1 = 0$ and $x_2 = 0$
(ii) $x_1 = 0$ and $7x_1 + 5x_2 = 4$
(iii) $x_2 = 0$ and $2x_1 + 2x_2 = 1$
(iv) $2x_1 + 2x_2 = 1$ and $7x_1 + 5x_2 = 4$

First, we will compute the equilibria in these four cases:

Case (i) There is nothing to compute; the equilibrium is $(\hat{x}_1, \hat{x}_2) = (0, 0)$.

Case (ii) To find the equilibrium, we must solve the system

$$\begin{aligned} x_1 &= 0 \\ 7x_1 + 5x_2 &= 4 \end{aligned}$$

which has the solutions

$$x_1 = 0 \qquad \text{and} \qquad x_2 = \frac{4}{5}$$

Hence, the equilibrium is $(\hat{x}_1, \hat{x}_2) = (0, \frac{4}{5})$.

Case (iii) To find the equilibrium, we must solve the system

$$\begin{aligned} x_2 &= 0 \\ 2x_1 + 2x_2 &= 1 \end{aligned}$$

which has the solutions

$$x_2 = 0 \qquad \text{and} \qquad x_1 = \frac{1}{2}$$

Hence, the equilibrium is $(\hat{x}_1, \hat{x}_2) = (\frac{1}{2}, 0)$.

Case (iv) To find the equilibrium, we must solve the system

$$\begin{aligned} 2x_1 + 2x_2 &= 1 \\ 7x_1 + 5x_2 &= 4 \end{aligned}$$

We use the standard elimination method: Leaving the first equation alone, changing the second by multiplying the first by 7 and the second by 2, and subtracting the second equation from the first, we find that this system is equivalent to

$$\begin{aligned} 2x_1 + 2x_2 &= 1 \\ 4x_2 &= -1 \end{aligned}$$

which has the solutions

$$x_2 = -\frac{1}{4} \qquad \text{and} \qquad x_1 = \frac{1}{2} - x_2 = \frac{3}{4}$$

Hence, the equilibrium is $(\hat{x}_1, \hat{x}_2) = (\frac{3}{4}, -\frac{1}{4})$.

We can illustrate all equilibria in the direction field of (11.54), which is displayed in Figure 11.36. The equilibria are shown as dots.

(b) To analyze the stability of the equilibria, we compute the Jacobi matrix

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

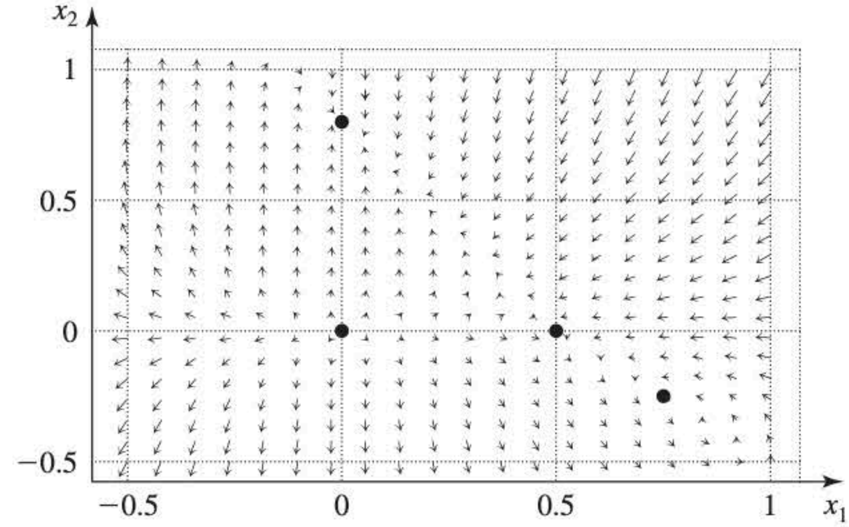

Figure 11.36 The direction field of (11.54) together with the equilibria.

With

$$f_1(x_1, x_2) = x_1 - 2x_1^2 - 2x_1x_2 \quad \text{and} \quad f_2(x_1, x_2) = 4x_2 - 5x_2^2 - 7x_1x_2$$

we find that

$$D\mathbf{f}(x_1, x_2) = \begin{bmatrix} 1 - 4x_1 - 2x_2 & -2x_1 \\ -7x_2 & 4 - 10x_2 - 7x_1 \end{bmatrix}$$

We will now go through the four cases and analyze each equilibrium:

Case (i) The equilibrium is the point (0, 0). The Jacobi matrix at (0, 0) is

$$D\mathbf{f}(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Since this matrix is in diagonal form, the eigenvalues are the diagonal elements, and we find that $\lambda_1 = 1$ and $\lambda_2 = 4$. Because both eigenvalues are positive, the equilibrium is unstable. Using the same classification as in the linear case, we say that (0, 0) is an unstable node.

The linearization of the direction field about (0, 0) is displayed in Figure 11.37, where we show the direction field of

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{x}(t)$$

Figure 11.37 confirms that (0, 0) is an unstable node (or source). If you now compare Figure 11.37 with the direction field of (11.54) shown in Figure 11.36,

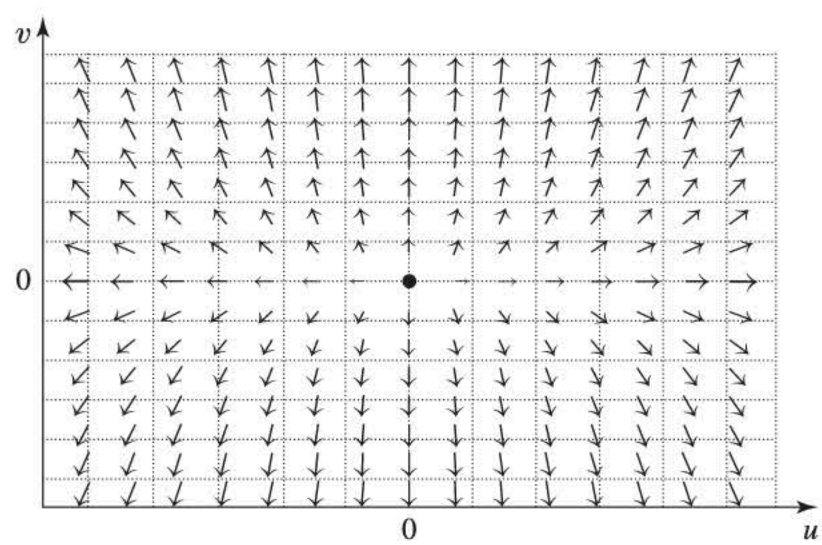

Figure 11.37 The linearization of the direction field about (0, 0).

you will find that the linearized direction field and the direction field of (11.54) close to the equilibrium (0, 0) are similar.

Case (ii) The equilibrium is the point $(0, \frac{4}{5})$. The Jacobi matrix at $(0, \frac{4}{5})$ is

$$\mathbf{Df}\left(0, \frac{4}{5}\right) = \begin{bmatrix} 1 - \frac{8}{5} & 0 \\ -\frac{28}{5} & 4 - 8 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & 0 \\ -\frac{28}{5} & -4 \end{bmatrix}$$

Since this matrix is in lower triangular form, the eigenvalues are the diagonal elements, and we find that the eigenvalues of $\mathbf{Df}(0, \frac{4}{5})$ are $\lambda_1 = -\frac{3}{5}$ and $\lambda_2 = -4$. Because both eigenvalues are negative, $(0, \frac{4}{5})$ is locally stable. Using the same classification as in the linear case, we say that $(0, \frac{4}{5})$ is a stable node.

The linearization of the direction field about $(0, \frac{4}{5})$ is displayed in Figure 11.38, which confirms that $(0, \frac{4}{5})$ is a stable node (or sink). If you compare Figure 11.38 with the direction field of (11.54) shown in Figure 11.36, you will find that the linearized direction field and the direction field of (11.54) close to the equilibrium $(0, \frac{4}{5})$ are similar.

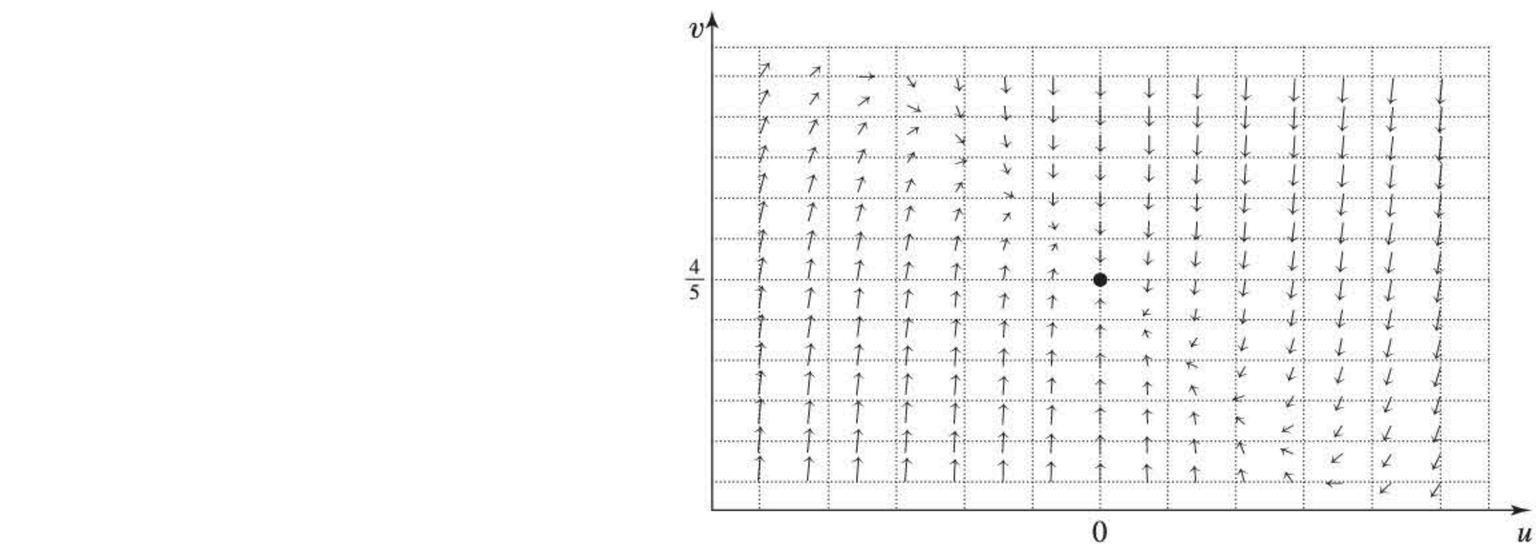

Figure 11.38 The linearization of the direction field about $(0, \frac{4}{5})$.

Case (iii) The equilibrium is the point $(\frac{1}{2}, 0)$. The Jacobi matrix at $(\frac{1}{2}, 0)$ is

$$\mathbf{Df}\left(\frac{1}{2}, 0\right) = \begin{bmatrix} 1 - 2 & -1 \\ 0 & 4 - \frac{7}{2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Since this matrix is in upper triangular form, the eigenvalues are simply the diagonal elements, and we find that the eigenvalues of $\mathbf{Df}(\frac{1}{2}, 0)$ are $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{2}$. Because one eigenvalue is positive and the other is negative, $(\frac{1}{2}, 0)$ is unstable. Using the same classification as in the linear case, we say that $(\frac{1}{2}, 0)$ is a saddle point.

The linearization of the direction field about $(\frac{1}{2}, 0)$ is displayed in Figure 11.39, which confirms that $(\frac{1}{2}, 0)$ is a saddle point. If you compare Figure 11.39 with the direction field of (11.54) shown in Figure 11.36, you will find that the linearized direction field and the direction field of (11.54) close to the equilibrium $(\frac{1}{2}, 0)$ are similar.

Case (iv) The Jacobi matrix at the equilibrium $(\frac{3}{4}, -\frac{1}{4})$ is

$$\mathbf{Df}\left(\frac{3}{4}, -\frac{1}{4}\right) = \begin{bmatrix} 1 - 3 + \frac{1}{2} & -\frac{3}{2} \\ \frac{7}{4} & 4 + \frac{10}{4} - \frac{21}{4} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ \frac{7}{4} & \frac{5}{4} \end{bmatrix}$$

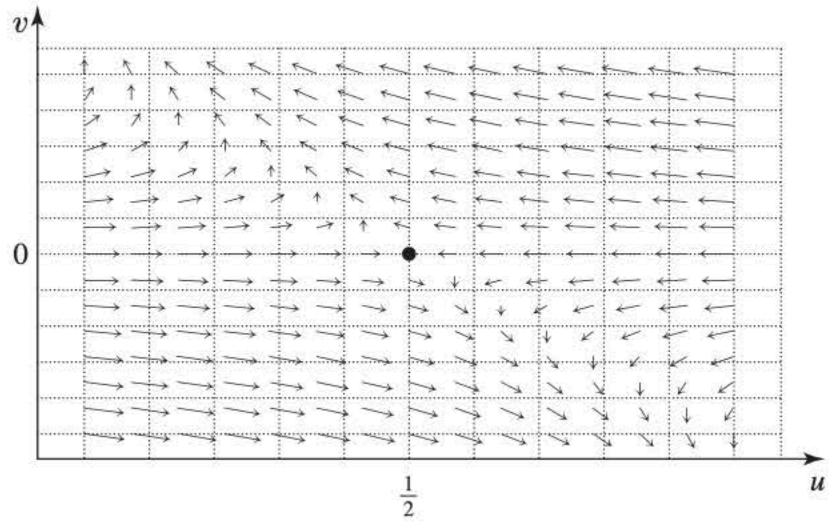

Figure 11.39 The linearization of the direction field about $(\frac{1}{2}, 0)$.

To find the eigenvalues, we must solve

$$\det\begin{bmatrix} -\frac{3}{2}-\lambda & -\frac{3}{2} \\ \frac{7}{4} & \frac{5}{4}-\lambda \end{bmatrix} = 0$$

Evaluating the determinant on the left-hand side and simplifying yields

$$\left(-\frac{3}{2}-\lambda\right)\left(\frac{5}{4}-\lambda\right)+\left(\frac{3}{2}\right)\left(\frac{7}{4}\right)=0$$

$$\lambda^2+\frac{3}{2}\lambda-\frac{5}{4}\lambda-\frac{15}{8}+\frac{21}{8}=0$$

$$\lambda^2+\frac{1}{4}\lambda+\frac{3}{4}=0$$

Solving this quadratic equation, we find that

$$\lambda_{1,2}=\frac{-\frac{1}{4}\pm\sqrt{\frac{1}{16}-3}}{2}$$

$$=-\frac{1}{8}\pm\frac{1}{8}\sqrt{-47}=-\frac{1}{8}\pm\frac{1}{8}i\sqrt{47}$$

That is,

$$\lambda_1=-\frac{1}{8}+\frac{1}{8}i\sqrt{47} \quad \text{and} \quad \lambda_2=-\frac{1}{8}-\frac{1}{8}i\sqrt{47}$$

The eigenvalues are complex conjugates with negative real parts. Thus, $(\frac{3}{4}, -\frac{1}{4})$ is locally stable, and we expect the solutions to spiral into the equilibrium when we start close to the equilibrium.

The linearization of the direction field about $(\frac{3}{4}, -\frac{1}{4})$ is displayed in Figure 11.40, which confirms that $(\frac{3}{4}, -\frac{1}{4})$ is a stable spiral. If you compare Figure 11.40 with the direction field of (11.54) shown in Figure 11.36, you will find that the linearized direction field and the direction field of (11.54) close to the equilibrium $(\frac{3}{4}, -\frac{1}{4})$ are similar. ■

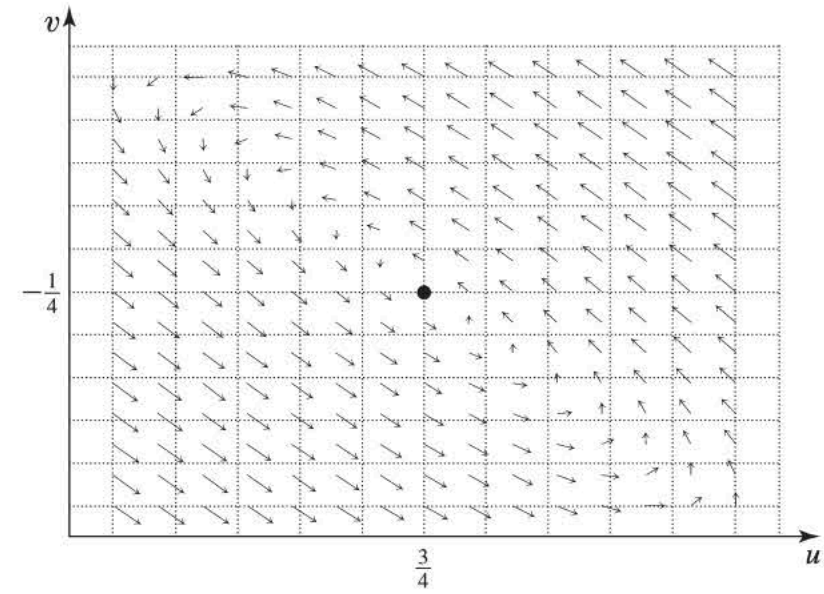

Figure 11.40 The linearization of the direction field about $(\frac{3}{4}, -\frac{1}{4})$.

■ 11.3.2 Graphical Approach for 2×2 Systems

In this subsection, we will discuss a graphical approach to systems of two autonomous differential equations. Suppose that

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

which in vector form is

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

The curves

$$f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$

are called **zero isoclines** or **null clines**, and they represent the points in the x_1–x_2 plane where the growth rates of the respective quantities are equal to zero. This situation is illustrated in Figure 11.41 for a particular choice of f_1 and f_2. Let's assume that x_1 and x_2 are nonnegative; this restricts the discussion to the first quadrant of the x_1–x_2 plane. The two curves in Figure 11.41 divide the first quadrant into four regions, and we label each region according to whether dx_i/dt is positive or negative. Without specifying the signs of f_1 and f_2 any further, we make assumptions about the signs of dx_1/dt and dx_2/dt as indicated in Figure 11.41.

The point where both null clines in Figure 11.41 intersect is a point equilibrium or critical point, which we call $\hat{\mathbf{x}}$. We can use the graph to determine the signs of the entries in the Jacobi matrix

$$D\mathbf{f}(\hat{\mathbf{x}}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_{ij} = \frac{\partial f_i}{\partial x_j}(\hat{\mathbf{x}})$. Clearly, the entry $a_{11} = \frac{\partial f_1}{\partial x_1}$ is the effect of a change in f_1 in the x_1-direction when we keep x_2 fixed. To determine the sign of a_{11}, follow the horizontal arrow in the figure: The arrow goes from a region where f_1 is positive to a region where f_1 is negative, implying that f_1 is decreasing and hence $\frac{\partial f_1}{\partial x_1}(\hat{\mathbf{x}}) = a_{11} < 0$. We conclude that the sign of a_{11} in $D\mathbf{f}(\hat{\mathbf{x}})$ is negative, which we indicate in the Jacobi matrix by a minus sign in place of a_{11}:

$$D\mathbf{f}(\hat{\mathbf{x}}) = \begin{bmatrix} - & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

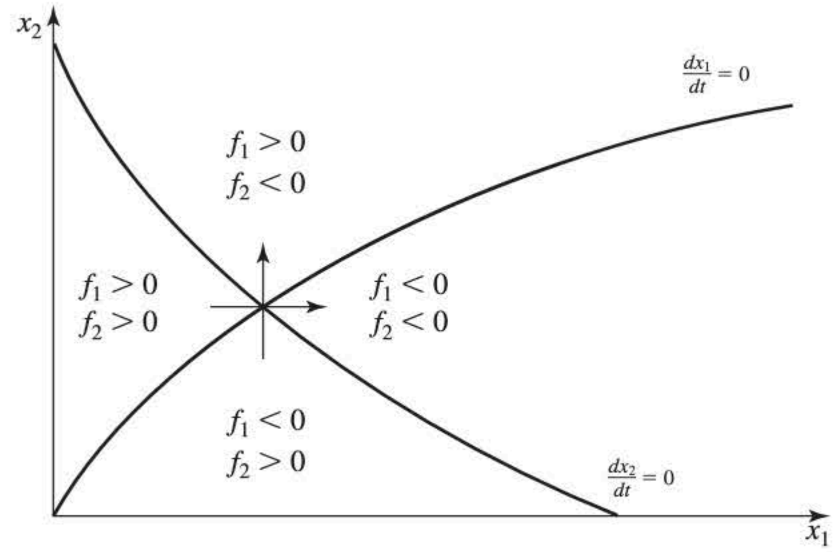

Figure 11.41 Graphical approach: zero isoclines.

Next, we determine the sign of $a_{12} = \frac{\partial f_1}{\partial x_2}(\hat{\mathbf{x}})$. This time, we want to know how f_1 changes at the equilibrium $\hat{\mathbf{x}}$ when we move in the x_2-direction and keep x_1 fixed. This is the direction of the vertical arrow through the equilibrium point. Since the arrow goes from a region where f_1 is negative to a region where f_1 is positive, f_1 increases in the direction of x_2 and, therefore, $a_{12} = \frac{\partial f_1}{\partial x_2}(\hat{\mathbf{x}}) > 0$.

The signs of the other two entries are found similarly, and we obtain

$$D\mathbf{f}(\hat{\mathbf{x}}) = \begin{bmatrix} - & + \\ - & - \end{bmatrix}$$

Thus, the trace of $D\mathbf{f}(\hat{\mathbf{x}})$ is negative and the determinant of $D\mathbf{f}(\hat{\mathbf{x}})$ is positive. Using the criterion stated in Subsection 9.4.2, we conclude that both eigenvalues have negative real parts and, therefore, that the equilibrium is locally stable.

EXAMPLE 3 Use the graphical approach to analyze the equilibrium (3, 2) of

$$\frac{dx_1}{dt} = 5 - x_1 - x_1x_2 + 2x_2$$

$$\frac{dx_2}{dt} = x_1x_2 - 3x_2$$

Solution First, note that (3, 2) is indeed an equilibrium of this system. Now, the zero isoclines satisfy

$$\frac{dx_1}{dt} = 0, \quad \text{which holds for } x_2 = \frac{5 - x_1}{x_1 - 2}$$

and

$$\frac{dx_2}{dt} = 0, \quad \text{which holds for } x_2 = 0 \text{ or } x_1 = 3$$

The zero isoclines in the x_1–x_2 plane are drawn in Figure 11.42. The equilibrium (3, 2) is the point of intersection of the zero isoclines $x_1 = 3$ and $x_2 = \frac{5-x_1}{x_1-2}$. The signs of $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ are indicated in the figure as well. We claim that

$$D\mathbf{f}(\hat{\mathbf{x}}) = \begin{bmatrix} - & - \\ + & 0 \end{bmatrix}$$

Here is why: To find the sign of $a_{11} = \frac{\partial f}{\partial x_1}$, we need to determine how dx_1/dt changes as we cross the zero isocline of x_1 in the x_1-direction. We see from the graph that dx_1/dt changes from positive to negative when we follow the horizontal arrow while crossing the zero isocline of x_1. Therefore, a_{11} is negative. To see why $a_{22} = 0$, follow

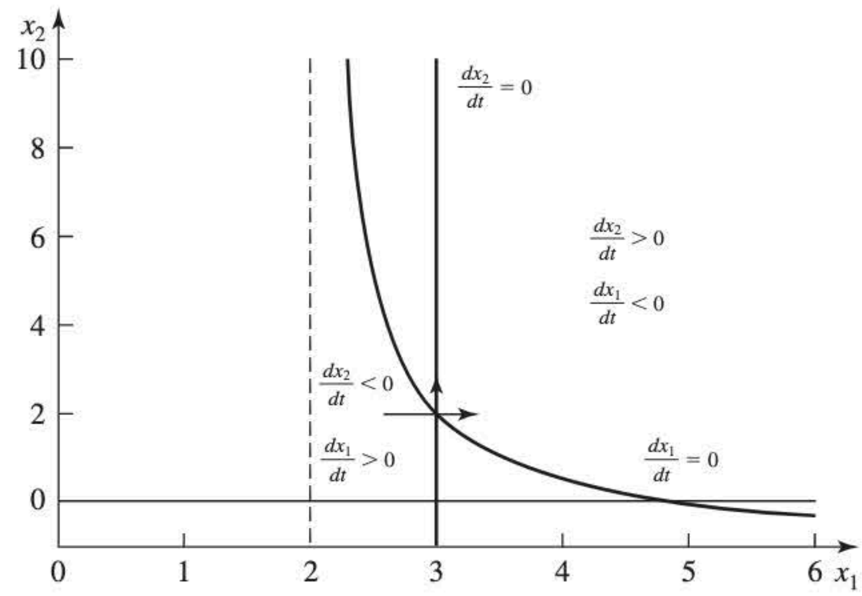

Figure 11.42 The zero isoclines in the x_1–x_2 plane.

the vertical arrow in the x_2-direction. Since the vertical arrow is on the zero isocline of x_2, the sign of dx_2/dt does not change as we cross the equilibrium in the x_2-direction. Therefore, $a_{22} = 0$. The signs of a_{12} and a_{21} follow from observing that if we cross the zero isocline of x_1 in the x_2-direction (the vertical arrow), then dx_1/dt changes from positive to negative, making $a_{12} < 0$. If we cross the zero isocline of x_2 in the direction of x_1 (the horizontal arrow), we see that dx_2/dt changes from negative to positive, making $a_{21} > 0$.

To determine the stability of $\hat{\mathbf{x}}$, we look at the trace and the determinant. Since the trace is negative and the determinant is positive, we conclude that the equilibrium is locally stable. ■

This simple graphical approach does not always give us the signs of the real parts of the eigenvalues, as illustrated in the following example: Suppose that we arrive at the Jacobi matrix in which the signs of the entries are

$$\begin{bmatrix} + & - \\ - & - \end{bmatrix}$$

The trace may now be positive or negative. Therefore, we cannot conclude anything about the eigenvalues. In this case, we would have to compute the eigenvalues or the trace and the determinant explicitly and cannot rely on the signs alone.

Section 11.3 Problems

■ 11.3.1

In Problems 1–6, the point $(0, 0)$ is always an equilibrium. Use the analytical approach to investigate its stability.

1. $$\frac{dx_1}{dt} = x_1 - 2x_2 + x_1x_2$$
$$\frac{dx_2}{dt} = -x_1 + x_2$$

2. $$\frac{dx_1}{dt} = -x_1 - x_2 + x_1^2$$
$$\frac{dx_2}{dt} = x_2 - x_1^2$$

3. $$\frac{dx_1}{dt} = x_1 + x_1^2 - 2x_1x_2 + x_2$$
$$\frac{dx_2}{dt} = x_1$$

4. $$\frac{dx_1}{dt} = 3x_1x_2 - x_1 + x_2$$
$$\frac{dx_2}{dt} = x_2^2 - x_1$$

5. $$\frac{dx_1}{dt} = x_1e^{-x_2}$$
$$\frac{dx_2}{dt} = 2x_2e^{x_1}$$

6. $$\frac{dx_1}{dt} = -2\sin x_1$$
$$\frac{dx_2}{dt} = -x_2e^{x_1}$$

In Problems 7–12, find all equilibria of each system of differential equations and use the analytical approach to determine the stability of each equilibrium.

7. $$\frac{dx_1}{dt} = -x_1 + 2x_1(1 - x_1)$$
$$\frac{dx_2}{dt} = -x_2 + 5x_2(1 - x_1 - x_2)$$

8. $$\frac{dx_1}{dt} = -x_1 + 3x_1(1 - x_1 - x_2)$$
$$\frac{dx_2}{dt} = -x_2 + 5x_2(1 - x_1 - x_2)$$

9. $$\frac{dx_1}{dt} = 4x_1(1 - x_1) - 2x_1x_2$$
$$\frac{dx_2}{dt} = x_2(2 - x_2) - x_2$$

10. $$\frac{dx_1}{dt} = 2x_1(5 - x_1 - x_2)$$
$$\frac{dx_2}{dt} = 3x_2(7 - 3x_1 - x_2)$$

11. $\frac{dx_1}{dt} = x_1 - x_2$

$\frac{dx_2}{dt} = x_1 x_2 - x_2$

12. $\frac{dx_1}{dt} = x_1 x_2 - x_2$

$\frac{dx_2}{dt} = x_1 + x_2$

13. For which value of a has

$$\frac{dx_1}{dt} = x_2(x_1 + a)$$
$$\frac{dx_2}{dt} = x_2^2 + x_2 - x_1$$

a unique equilibrium? Characterize its stability.

14. Assume that $a > 0$. Find all point equilibria of

$$\frac{dx_1}{dt} = 1 - a x_1 x_2$$
$$\frac{dx_2}{dt} = a x_1 x_2 - x_2$$

and characterize their stability.

■ 11.3.2

15. Assume that

$$\frac{dx_1}{dt} = x_1(10 - 2x_1 - x_2)$$
$$\frac{dx_2}{dt} = x_2(10 - x_1 - 2x_2)$$

(a) Graph the zero isoclines.

(b) Show that $(\frac{10}{3}, \frac{10}{3})$ is an equilibrium, and use the analytical approach to determine its stability.

16. Assume that

$$\frac{dx_1}{dt} = x_1(10 - x_1 - 2x_2)$$
$$\frac{dx_2}{dt} = x_2(10 - 2x_1 - x_2)$$

(a) Graph the zero isoclines.

(b) Show that $(\frac{10}{3}, \frac{10}{3})$ is an equilibrium, and use the analytical approach to determine its stability.

In Problems 17–22, use the graphical approach for 2×2 systems to find the sign structure of the Jacobi matrix at the indicated equilibrium. If possible, determine the stability of the equilibrium. Assume that the system of differential equations is given by

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

Furthermore, assume that x_1 and x_2 are both nonnegative. In each problem, the zero isoclines are drawn and the equilibrium we want to investigate is indicated by a dot. Assume that both x_1 and x_2 increase close to the origin and that f_1 and f_2 change sign when crossing their zero isoclines.

17. See Figure 11.43.

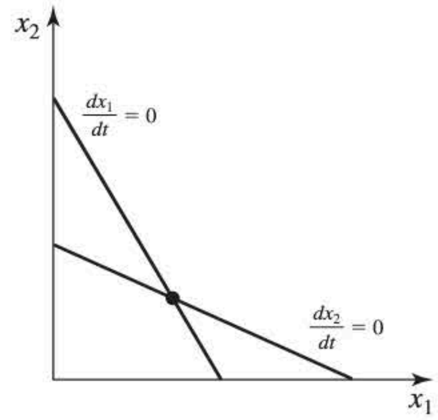

Figure 11.43

18. See Figure 11.44.

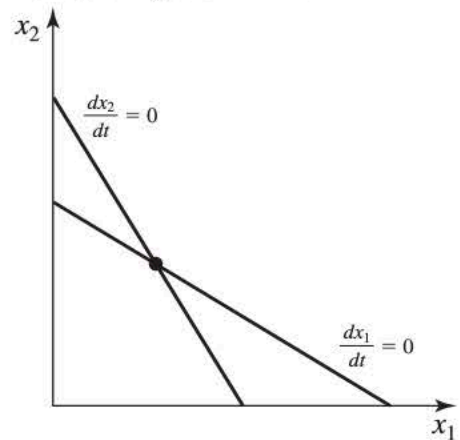

Figure 11.44

19. See Figure 11.45.

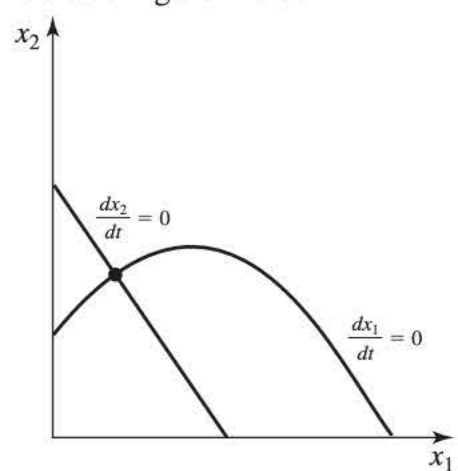

Figure 11.45

20. See Figure 11.46.

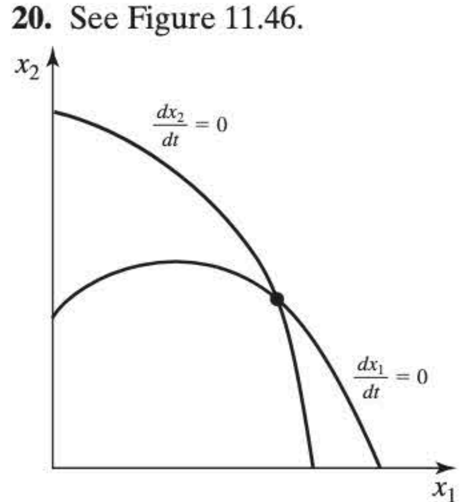

Figure 11.46

21. See Figure 11.47.

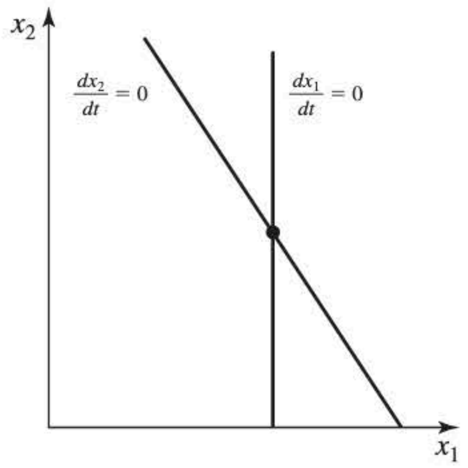

Figure 11.47

22. See Figure 11.48.

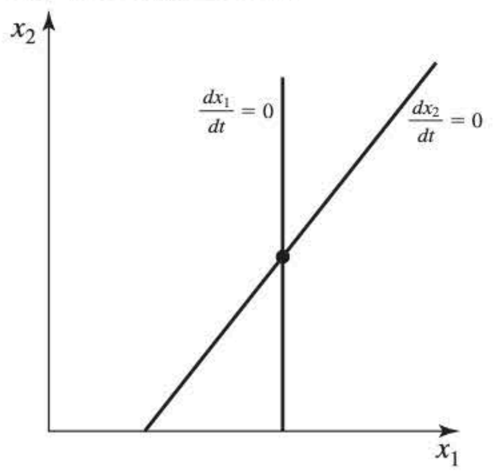

Figure 11.48

23. Let

$$\frac{dx_1}{dt} = x_1(2 - x_1) - x_1x_2$$
$$\frac{dx_2}{dt} = x_1x_2 - x_2$$

(a) Graph the zero isoclines.

(b) Show that (1, 1) is an equilibrium. Use the graphical approach to determine its stability.

24. Let

$$\frac{dx_1}{dt} = x_1(2 - x_1^2) - x_1x_2$$
$$\frac{dx_2}{dt} = x_1x_2 - x_2$$

(a) Graph the zero isoclines.

(b) Show that (1, 1) is an equilibrium. Use the graphical approach to determine its stability.

11.4 Nonlinear Systems: Applications

11.4.1 The Lotka–Volterra Model of Interspecific Competition

Imagine two species of plants growing together in the same plot. They both use similar resources: light, water, and nutrients. The use of these resources by one individual reduces their availability to other individuals. We call this type of interaction between individuals **competition**. **Intraspecific competition** occurs between individuals of the same species, **interspecific competition** between individuals of different species. Competition may result in reduced fecundity or reduced survivorship (or both). The effects of competition are often more pronounced when the number of competitors is higher.

In this subsection, we will discuss the Lotka–Volterra model of interspecific competition, which incorporates density-dependent effects of competition in the manner described previously. The model is an extension of the logistic equation to the case of two species. To describe it, we denote the population size of species 1 at time t by $N_1(t)$ and that of species 2 at time t by $N_2(t)$. Each species grows according to the logistic equation when the other species is absent. We denote their respective carrying capacities by K_1 and K_2, and their respective intrinsic rates of growth by r_1 and r_2. We assume that K_1, K_2, r_1, and r_2 are positive. In addition, the two species may have inhibitory effects on each other. We measure the effect of species 1 on species 2 by the **competition coefficient** α_{21}; the effect of species 2 on species 1 is measured by the competition coefficient α_{12}. The Lotka–Volterra model of interspecific competition is then given by the following system of differential