MA 138 – Calculus 2 with Life Science Applications

Section 6.3 (Applications of integration)

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Section 6.3: Applications of Integration

We are interested in the following three applications of integrals:

(1) **average** of a continuous function on $[a, b]$;

(2) **area between curves**;

(3) **cumulative change**.

Average Values

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \dots, y_n$:

$$y_{\text{avg}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

In general, let's try to compute the average value of a function $y = f(x)$, $a \le x \le b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $c_1, \dots, c_n$ in successive subintervals and calculate the average of the numbers $f(c_1), \dots, f(c_n)$:

$$\frac{f(c_1) + \cdots + f(c_n)}{n}$$

Since $\Delta x = (b - a)/n$, we can write $1/n = \Delta x/(b - a)$ and the average value becomes

$$\frac{f(c_1)\Delta x + \cdots + f(c_n)\Delta x}{b - a} = \frac{1}{b - a}\sum_{i=1}^{n} f(c_i)\Delta x.$$

If we let n increase, we would be computing the average value of a large number of closely spaced values. More precisely,

$$\lim_{n\to\infty} \frac{1}{b - a}\sum_{i=1}^{n} f(c_i)\Delta x = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Average of a Continuous Function on $[a, b]$

Assume that $f(x)$ is a continuous function on $[a, b]$. The average value of f on the interval $[a, b]$ is defined to be

$$f_{\text{avg}} = \frac{1}{b - a}\int_a^b f(x)\,dx,$$

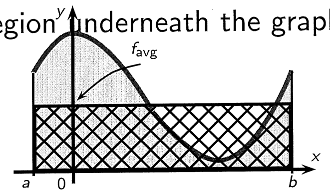
Geometric Meaning

Mean Value Theorem for Definite Integrals

Assume that $f(x)$ is a continuous function on $[a, b]$. Then there exists a number $c \in [a, b]$ such that

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

That is, when f is continuous, there exists a number c such that $f(c) = f_{\text{avg}}$. If f is a continuous, positive valued function, f_{avg} is that number such that the rectangle with base $[a, b]$ and height f_{avg} has the same area as the region underneath the graph of f from a to b.



Course Introduction Section 6.3 Average Area Between Curves Cumulative Change

Example 1 (Online Homework #14)

If a cup of coffee has temperature 95°C in a room where the temperature is 20°C, then, according to Newton's Law of Cooling, the temperature of the coffee after t minutes is

$$T(t) = 20 + 75e^{-t/50}.$$

What is the average temperature (in degrees Celsius) of the coffee during the first half hour?

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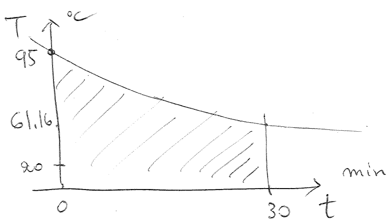
Notice that the graph of $T(t) = 20 + 75e^{-t/50}$ looks like:



We want to compute

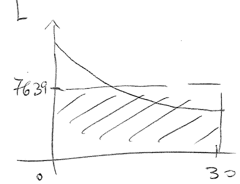
$$T_{avg} = \frac{1}{30-0} \cdot \int_0^{30} \left(20 + 75e^{-t/50}\right) dt$$

$$= \frac{1}{30}\left[20t + 75e^{-t/50} \cdot (-50)\right]_0^{30} =$$

$$= \frac{1}{30}\left[\left(20 \cdot 30 - 3750e^{-30/50}\right) - \left(0 - 3750\right)\right]$$

$$= \frac{1}{30}\left[600 - 3750\left(e^{-3/5} - 1\right)\right] = \frac{1}{30}\left[4350 - 3750e^{-3/5}\right]$$

$$= 145 - 125e^{-3/5} \approx \boxed{76.39\ ^\circ C}$$

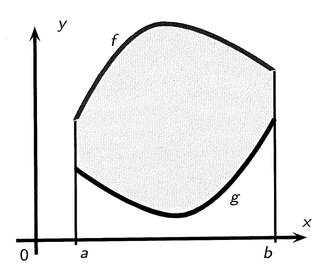


Course Introduction Section 6.3 Average Area Between Curves Cumulative Change

Area Between Curves

Assume f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$. The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines$x = a$, $x = b$, is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



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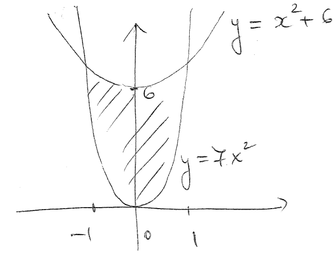
Example 2 (Online Homework #2)

Find the area of the region enclosed by the two functions $y = 7x^2$ and $y = x^2 + 6$.



first of all we need to find the intersection points of

$$y = 7x^2 \quad \text{and} \quad y = x^2 + 6$$

$$x^2 + 6 = 7x^2 \iff 6x^2 = 6 \iff x^2 = 1 \iff x = \pm 1$$

Thus the area we seek is

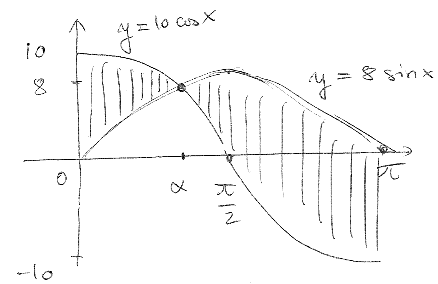
$$\int_{-1}^{1} \left[(x^2+6) - (7x^2) \right] dx = \int_{-1}^{1} (6 - 6x^2)\, dx =$$

$$= \text{by symmetry} = 2 \int_0^1 6(1-x^2)\, dx = \left[2(6x - 2x^3) \right]_0^1$$

$$= \left[(12 - 4) - (0) \right] = \boxed{8}$$

Example 3 (Online Homework #3)

Find the area between $y = 8 \sin x$ and $y = 10 \cos x$ over the interval $[0, \pi]$. Sketch the curves if necessary.



α is the angle such that

$$10 \cos\alpha = y = 8 \sin\alpha$$

OR $\quad \dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{10}{8} = \boxed{\dfrac{5}{4}}$

Thus the area we want is:

$$\int_0^{\alpha} (10\cos x - 8 \sin x) + \int_{\alpha}^{\pi} (8 \sin x - 10 \cos x)\, dx =$$

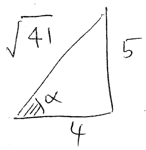
$$= \left[10 \sin x + 8 \cos x \right]_0^{\alpha} + \left[-8 \cos x - 10 \sin x \right]_{\alpha}^{\pi} =$$

$$= (10 \sin\alpha + 8 \cos\alpha \cancel{-8}) + (\cancel{8} + 8 \cos\alpha + 10 \sin\alpha)$$

$$= 20 \sin\alpha + 16 \cos\alpha$$

$$= 20 \sin\alpha + 16\cos\alpha$$

Now $\tan\alpha = \frac{10}{8} = \frac{5}{4}$ so



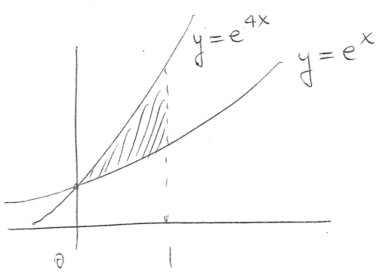
$\cos\alpha = \frac{4}{\sqrt{41}}$ $\sin\alpha = \frac{5}{\sqrt{41}}$

$$\therefore \text{Area} = 20\cdot\frac{5}{\sqrt{41}} + 16\cdot\frac{4}{\sqrt{41}} = \frac{164}{\sqrt{41}} = \frac{4\cdot 41}{\sqrt{41}}$$

$$= \boxed{4\sqrt{41}} \approx 25.6125$$

Example 4 (Online Homework #4)

Find the area between $y = e^x$ and $y = e^{4x}$ over $[0, 1]$.



$$\text{Area} = \int_0^1 (e^{4x} - e^x)\,dx = \left.\frac{1}{4}e^{4x} - e^x\right]_0^1 =$$

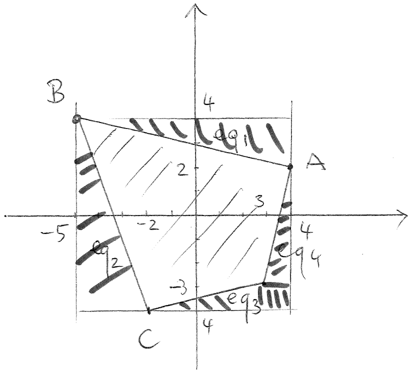
$$= \left[\frac{1}{4}e^4 - e\right] - \left[\frac{1}{4}e^0 - e^0\right]$$

$$= \frac{1}{4}e^4 - e - \left(\frac{1}{4} - 1\right) = \boxed{\frac{1}{4}e^4 - e + \frac{3}{4}}$$

$$= \frac{e^4 - 4e + 3}{4} \cong \underline{11.6813}$$

Example 5 (Online Homework #6)

Find the area of the quadrangle with vertices $(4, 2)$, $(-5, 4)$, $(-2, -4)$, and $(3, -3)$.



A (4,2)

B (−5,4)

C (−2,−4)

D (3,−3)

One can certainly compute the equations of the 4 lines and do: $\int_{-5}^{-2}(eq_1 - eq_2)\,dx + \int_{-2}^{3}(eq_1 - eq_3)\,dx + \dots$

OR $8\cdot 9 = 72$ = area of the big rectangle − minus the area of the 4 triangles and one square

i.e. $72 - \left(\frac{8\cdot 3}{2} + \frac{5\cdot 1}{2} + \frac{5\cdot 1}{2} + \frac{9\cdot 2}{2} + 1\right)$

$= 72 - 12 - 5 - 9 - 1 = 72 - 27 = \boxed{45}$

Example 6 (Online Homework #7)

Consider the area between the graphs $x + y = 14$ and $x + 6 = y^2$.

This area can be computed in two different ways using integrals.

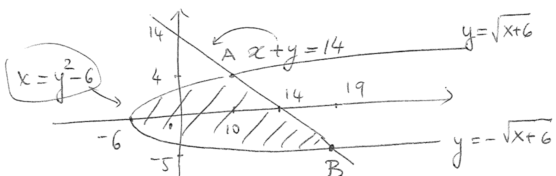
- First of all it can be computed as a sum of two integrals
$$\int_a^b f(x)\,dx + \int_b^c g(x)\,dx$$
where $a =$ ____, $b =$ ____, $c =$ ____, and $f(x) =$ ____ $g(x) =$ ____.
- Alternatively this area can be computed as a single integral
$$\int_\alpha^\beta h(y)\,dy$$
where $\alpha =$ ____, $\beta =$ ____, and $h(y) =$ ____.

The intersection points of $\boxed{x+y=14}$ and $\boxed{x+6=y^2}$ are given by

$\begin{cases} x+y=14 \\ x+6=y^2 \end{cases} \iff 14-y = x = y^2-6 \iff y^2+y-20=0$

$\iff (y+5)(y-4)=0 \quad \therefore y = 4, -5$ hence $x = 10, 19$

$\therefore$ A(10, 4) B(19, −5). The graph is:



We can compute the area in 2 ways:

① $\int_{-6}^{10}\left[\sqrt{x+6} - (-\sqrt{x+6})\right]dx + \int_{10}^{19}\left[(14-x) - (-\sqrt{x+6})\right]dx$

$= \int_{-6}^{10} 2\sqrt{x+6}\,dx + \int_{10}^{19}(14 - x + \sqrt{x+6})\,dx =$

$= \left[\frac{4}{3}(x+6)^{3/2}\right]_{-6}^{10} + \left[14x - \frac{1}{2}x^2 + \frac{2}{3}(x+6)^{3/2}\right]_{10}^{19} =$

$= \left[\frac{4}{3}\cdot 64 - 0\right] + \left[\left(14\cdot 19 - \frac{19^2}{2} + \frac{2}{3}\cdot 125\right) - \left(140 - 50 + \frac{2}{3}\cdot 64\right)\right]$

$= \frac{2}{3}\cdot 64 + 266 - \frac{361}{2} + 125\,\frac{2}{3} - 90$

$= \frac{2}{\cancel{3}}\left(\cancel{189}^{\,63}\right) + 266 - 180 - \frac{1}{2} - 90 = 126 + 266 - 270 - \frac{1}{2}$

$= 392 - 270 - \frac{1}{2} = 122 - \frac{1}{2} = \underline{\underline{121.5}}$

Second way:

$\int_{-5}^{4}\left[(14-y) - (y^2-6)\right]dy = \int_{-5}^{4}(20 - y - y^2)\,dy =$

$= \left[20y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-5}^{4} = \left[\left(20\cdot 4 - \frac{1}{2}(4)^2 - \frac{1}{3}(4)^3\right) - \left(20(-5) - \frac{1}{2}(-5)^2 - \frac{1}{3}(-5)^3\right)\right] = \left[\left(80 - 8 - \frac{64}{3}\right) - \left(-100 - \frac{25}{2} + \frac{125}{3}\right)\right]$

$= 80 - 8 + 100 - \frac{64}{3} + \frac{25}{2} - \frac{125}{3} = 172 - \frac{\cancel{189}^{\,63}}{\cancel{3}} + \frac{25}{2} = 109 + 12.5 = \underline{\underline{121.5}}$

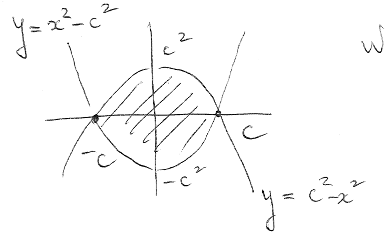
Example 7 (Online Homework #5)

Find the value(s) of c such that the area of the region bounded by the parabolae $y = x^2 - c^2$ and $y = c^2 - x^2$ is 1944.

$y = x^2 - c^2$ and $y = c^2 - x^2$ intersect at $x^2 - c^2 = c^2 - x^2$

$\iff 2x^2 = 2c^2 \iff x^2 = c^2$

OR $x = \pm c$. Their graph is



We want

$$\boxed{\int_{-c}^{c} \left[(c^2-x^2)-(x^2-c^2)\right] dx = 1944}$$

$$\int_{-c}^{c} (2c^2 - 2x^2)\, dx = 1944 \quad \text{by symmetry}$$

$$2\int_{0}^{c} (2c^2 - 2x^2)\, dx = 1944$$

$$\iff \int_0^c (c^2 - x^2)\, dx = \frac{1944}{4} = 486$$

$$\therefore \int_0^c (c^2 - x^2)\, dx = 486$$

$$c^2 x - \frac{1}{3}x^3 \Big]_0^c = 486$$

$$\iff c^3 - \frac{1}{3}c^3 - (0) = 486$$

$$\iff \frac{2}{3}c^3 = 486$$

$$\iff c^3 = 3 \cdot 243$$

$$c^3 = 729$$

$\therefore \boxed{c = 9}$ But notice that $\boxed{c = -9}$ also works!

$y = c^2 - x^2$; $y = x^2 - c^2$

Cumulative Change

Suppose that we have a population whose size at time t is given by $N(t)$. Suppose further that its rate of growth is given by the initial value problem

IVP: $\dfrac{dN}{dt} = f(t) \qquad N(0) = N_0.$

Then, by Part I of the Fundamental Theorem of Calculus we have that

$$N(t) = \int_0^t f(u)\, du + C$$

represents all antiderivatives of $f(t)$ [or dN/dt].

Now, $N(0) = \underbrace{\int_0^0 f(u)\, du}_{=0} + C = C$ so $C = N_0 = N(0)$. Therefore

$$N(t) = \int_0^t f(u)\, du + N_0 \quad \text{or} \quad \boxed{N(t) - N(0) = \int_0^t f(u)\, du}.$$

More generally, the IVP: $\frac{dN}{dt} = f(t) \qquad N(a) = N_a$ has solution

$$\boxed{N(t) - N(a) = \int_a^t f(u)\,du = \int_a^t \frac{dN}{du}\,du}.$$

That is

$$\left\{ \begin{array}{c} \text{cumulative change} \\ \text{on the interval } [a,t] \end{array} \right\} = \int_a^t \left\{ \begin{array}{c} \text{instantaneous rate of} \\ \text{change at time } u \end{array} \right\} du$$

Similarly, if $p(t)$ is the position function of an object at time t, then

$$\frac{dp}{dt} = v(t) \qquad p(a) = p_a$$

gives $\rightsquigarrow$ $$\boxed{\underbrace{p(b) - p(a)}_{\text{distance traveled on } [a,b]} = \int_a^b v(t)\,dt = \int_a^b \frac{dp}{dt}\,dt}.$$

Example 8 (Problem #2, Section 6.3, page 349)

Suppose the change in biomass $B(t)$ at time t during the interval $[0,12]$ follows the equation

$$\frac{dB}{dt} = \cos\left(\frac{\pi}{6}t\right).$$

How does the biomass at time $t = 12$ compare to the biomass at time $t = 0$?

$$\frac{dB}{dt} = \cos\left(\frac{\pi}{6}t\right)$$

Thus

$$B(12) - B(0) = \int_0^{12} \frac{dB}{dt}\,dt = \int_0^{12} \cos\left(\frac{\pi}{6}t\right)dt$$

$$= \frac{6}{\pi} \sin\left(\frac{\pi}{6}t\right)\Big]_0^{12}$$

$$= \frac{6}{\pi}\cdot\left(\sin\left(\frac{\pi}{6}\cdot 12\right) - \sin\left(\frac{\pi}{6}\cdot 0\right)\right)$$

$$= \frac{6}{\pi}\cdot\left(\sin(2\pi) - \sin(0)\right) = 0$$

Thus $\boxed{B(12) - B(0) = 0}$ OR $\underline{B(12) = B(0)}$

There is no change in biomass

Example 9 (Problem #6, Section 6.3, page 349)

If $\frac{dw}{dx}$ represents the rate of change of the weight of an organism of age x, explain what

$$\int_3^5 \frac{dw}{dx}\,dx$$

means.

$$\int_3^5 \frac{dw}{dx}\,dx = w(5) - w(3)$$

i.e. it represents the change in weight between age 3 and 5