

MA 138 – Calculus 2 with Life Science Applications

The Substitution Rule

(Section 7.1)

Alberto Corso
(alberto.corso@uky.edu)

Department of Mathematics
University of Kentucky

Section 7.1: The Substitution Rule

The substitution rule is the chain rule in integral form.

We therefore begin by recalling the chain rule.

Suppose that we wish to differentiate

$$f(x) = (6x^2 + 3)^3.$$

This is clearly a situation in which we need to use the chain rule.

We set $u = 6x^2 + 3$ so that $f(u) = u^3$.

The chain rule, using Leibniz notation, tells us that

$$\frac{df}{dx} = \frac{df}{du} \frac{du}{dx} = 3u^2 \cdot (6 \cdot 2x) = 3(6x^2 + 3)^2(12x).$$

Reversing these steps and integrating along the way, we get

$$\int 3(6x^2 + 3)^2(12x) dx = \int 3u^2 du = u^3 + C = (6x^2 + 3)^3 + C.$$

In the first step, we substituted u for $6x^2 + 3$ and used $du = 12x dx$.

This substitution simplified the integrand.

At the end, we substitute back $6x^2 + 3$ for u to get the final answer in terms of x .

We summarize this discussion, by stating the following **general principle**:

Substitution Rule for Indefinite Integrals

If $u = g(x)$, then
$$\int f(\underbrace{g(x)}_u) \underbrace{g'(x)}_{du} dx = \int f(u) du.$$

Example 1

Evaluate the indefinite integral $\int \cos x \sin x dx$

- by using the substitution $u = \cos x$;
- by using the substitution $u = \sin x$;
- by using the trigonometric identity $\sin(2x) = 2 \sin x \cos x$.

Compare your answers.

Consider the indefinite integral $\int \cos x \cdot \sin x \, dx$.

(I) If we set $u = \cos x$ then $\frac{du}{dx} = -\sin x$ so that

$$\sin x \cdot dx = -du$$

$$\begin{aligned} \text{Thus: } \int \underbrace{\cos x}_u \cdot \underbrace{\sin x \, dx}_{-du} &= \int -u \, du = -\int u \, du = -\frac{1}{2}u^2 + C \\ &= -\frac{1}{2}(\cos x)^2 + C_1 = \underline{\underline{-\frac{1}{2}\cos^2 x + C_1}} \end{aligned}$$

(II) If we set $u = \sin x$ then $\frac{du}{dx} = \cos x$ so that

$$\cos x \, dx = du$$

$$\begin{aligned} \text{Thus: } \int \underbrace{\cos x}_{du} \cdot \underbrace{\sin x \, dx}_u &= \int u \, du = \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(\sin x)^2 + C_2 = \underline{\underline{\frac{1}{2}\sin^2 x + C_2}} \end{aligned}$$

Notice that the 2 answers are consistent as we know that there is the trigonometric identity

$$\cos^2 x + \sin^2 x = 1 \quad \text{Thus}$$

$$-\frac{1}{2}\cos^2 x + C_1 = -\frac{1}{2}(1 - \sin^2 x) + C_1 = \frac{1}{2}\sin^2 x + \underbrace{C_1 - \frac{1}{2}}_{C_2}$$

first solution

(III) Recall the trig. identity $\cos x \sin x = \frac{1}{2} \sin(2x)$. Thus

$$\begin{aligned} \int \cos x \sin x \, dx &= \int \frac{1}{2} \sin(2x) \, dx \quad \text{Set } \underline{u = 2x} \text{ so that } \underline{du = 2 \, dx} \\ &= \int \frac{1}{2} \sin(u) \cdot \frac{1}{2} du = \int \frac{1}{4} \sin(u) \, du = \frac{1}{4} \int \sin(u) \, du \\ &= \frac{1}{4}(-\cos(u)) + C_3 = \underline{\underline{-\frac{1}{4}\cos(2x) + C_3}} \end{aligned}$$

This answer is consistent with the others: $\cos(2x) = \cos^2 x - \sin^2 x$

The Substitution Rule

Example 2

Evaluate the indefinite integral $\int (2x+1)e^{x^2+x} \, dx$.

$$\int (2x+1) e^{x^2+x} \, dx$$

Set $u = x^2 + x$ so that $\frac{du}{dx} = 2x + 1$;

hence $(2x+1) \, dx = du$.

$$\begin{aligned} \text{Thus } \int (2x+1) e^{x^2+x} \, dx &= \int e^u \, du = \\ &= e^u + C = \boxed{e^{x^2+x} + C} \\ &\quad \text{substitute back} \end{aligned}$$

Substitution Rule for Definite Integrals

Part II of the FTC says that when we evaluate a definite integral, we must find an antiderivative of the integrand and then evaluate the antiderivative at the limits of integration.

When we use the substitution $u = g(x)$ to find an antiderivative of an integrand, the antiderivative will be given in terms of u at first.

To complete the calculation, we can proceed in either of two ways:

- (1) we can leave the antiderivative in terms of u and change the limits of integration according to $u = g(x)$;
- (2) we can substitute $g(x)$ for u in the antiderivative and then evaluate the antiderivative at the limits of integration in terms of x .

Substitution Rule for Definite Integrals

The first method (1) is the more common one, and we summarize the procedure as follows:

Substitution Rule for Definite Integrals

$$\text{If } u = g(x), \text{ then } \int_a^b \underbrace{f[g(x)]}_u \underbrace{g'(x) dx}_{du} = \int_{g(a)}^{g(b)} f(u) du.$$

Example 3 (Online Homework # 6)

Evaluate the definite integral $\int_1^{e^5} \frac{dx}{x(1+\ln x)}$.

$$\int_1^{e^5} \frac{dx}{x(1+\ln x)} = \text{rewrite it as} = \int_1^{e^5} \frac{1}{(1+\ln x)} \cdot \frac{1}{x} dx$$

and notice that if we set $u = 1 + \ln x$ then $\frac{du}{dx} = \frac{1}{x}$.

Thus the integral becomes

$$\int \underbrace{\frac{1}{(1+\ln x)}}_{\frac{1}{u}} \cdot \underbrace{\frac{1}{x} dx}_{du} = \int \frac{1}{u} du. \quad \text{We also need to change the limits of integration!}$$

$$x=1 \rightsquigarrow u=1+\ln x \rightsquigarrow u=1+\ln(1)=1$$

$$x=e^5 \rightsquigarrow u=1+\ln x \rightsquigarrow u=1+\ln(e^5)=1+5=6$$

$$\therefore \int_1^6 \frac{1}{u} \cdot du = \ln|u| \Big|_1^6 = \ln(6) - \underbrace{\ln(1)}_{=0} = \boxed{\ln(6)} \approx 1.79176$$

Example 4 (Online Homework # 8)

Consider the indefinite integral $\int \frac{3}{3+e^x} dx$.

- The most appropriate substitution to simplify this integral is $u = f(x)$ where $f(x) = \underline{\hspace{2cm}}$.
We then have $dx = g(u)du$ where $g(u) = \underline{\hspace{2cm}}$.
(Hint: you need to back substitute for x in terms of u for this part.)
- After substituting into the original integral we obtain $\int h(u) du$ where $h(u) = \underline{\hspace{2cm}}$.
- To evaluate this integral rewrite the numerator as $3 = u - (u - 3)$.
Simplify, then integrate, thus obtaining $\int h(u) du = H(u)$ where $H(u) = \underline{\hspace{2cm}} + C$.

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Lecture 3

Example 4, cont.ed (Online Homework # 8)

- After substituting back for u we obtain our final answer

$$\int \frac{3}{3+e^x} dx = \underline{\hspace{2cm}} + C.$$

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Lecture 3

1) Consider $\int \frac{3}{3+e^x} dx$ we try the following substitution
 $u = 3+e^x$. (This is our $f(x)$.) Thus $\frac{du}{dx} = e^x$ or
 $du = e^x dx$. Notice that $e^x = u-3$ so that
 \rightarrow becomes $du = (u-3) dx \rightsquigarrow \boxed{dx = \frac{1}{u-3} du}$
 Thus $g(u) = \frac{1}{u-3}$.

2) After substituting we obtain

$$\int \frac{3}{3+e^x} dx = \int \frac{3}{u} \cdot \frac{1}{(u-3)} \cdot du = \int \underbrace{\frac{3}{u(u-3)}}_{h(u)} du$$

3) We are suggested to rewrite the numerator as
 $3 = u - (u-3)$

$$\begin{aligned} \text{Therefore } \int \frac{3}{u(u-3)} du &= \int \frac{u - (u-3)}{u(u-3)} du = \\ &= \int \left[\frac{u}{u(u-3)} - \frac{u-3}{u(u-3)} \right] du = \int \left[\frac{1}{u-3} - \frac{1}{u} \right] du \\ &= \ln|u-3| - \ln|u| + C = \underbrace{\ln \left| \frac{u-3}{u} \right|}_{H(u)} + C \end{aligned}$$

4) after substituting back $u = 3+e^x$, we obtain

$$\int \frac{3}{3+e^x} dx = \boxed{\ln \left| \frac{e^x}{3+e^x} \right| + C}$$

$$\underline{\underline{\text{or}}} \quad \underline{\underline{x - \ln(3+e^x) + C}}$$

Example 5 (Online Homework # 9)

Consider the definite integral $\int_0^1 x^2 \sqrt{5x+6} dx$.

- Then the most appropriate substitution to simplify this integral is $u = \underline{\hspace{2cm}}$. Then $dx = f(x)du$ where $f(x) = \underline{\hspace{2cm}}$.
- After making the substitution and simplifying we obtain the integral

$$\int_a^b g(u) du$$

where $g(u) = \underline{\hspace{2cm}}$, $a = \underline{\hspace{1cm}}$ and $b = \underline{\hspace{1cm}}$.

- This definite integral has value = $\underline{\hspace{2cm}}$.

(1) Consider $\int_0^1 x^2 \sqrt{5x+6} dx$. The most appropriate substitution seems to be $u = 5x+6$ so that $\frac{du}{dx} = 5$ or $\boxed{dx = \frac{1}{5} du}$. For the substitution notice that we also need to write x in terms of u !
 $u = 5x+6 \rightsquigarrow x = \frac{u-6}{5}$. Thus

$$\int_0^1 x^2 \sqrt{5x+6} dx = \int_6^{11} \left(\frac{u-6}{5}\right)^2 \sqrt{u} \cdot \left(\frac{1}{5} du\right)$$

$\left. \begin{array}{l} u = 5x+6 \\ dx = \frac{1}{5} du \\ x = \frac{u-6}{5} \end{array} \right\}$

$\left. \begin{array}{l} x=1 \rightsquigarrow u=5x+6 \rightsquigarrow u=11 \\ x=0 \rightsquigarrow u=5x+6 \rightsquigarrow u=6 \end{array} \right\}$ for the limits of integration

(2) So, after making the substitutions we obtain:

$$\int_6^{11} \left(\frac{u-6}{5}\right)^2 \cdot \sqrt{u} \cdot \left(\frac{1}{5} du\right) = \int_6^{11} \frac{1}{125} (u-6)^2 \cdot \sqrt{u} \cdot du$$

$$= \int_6^{11} \frac{1}{125} (u^2 - 12u + 36) \sqrt{u} \cdot du = \int_6^{11} \frac{1}{125} \underbrace{(u^{5/2} - 12u^{3/2} + 36u^{1/2})}_{g(u)} du$$

multiply out 6

(3) $= \frac{1}{125} \left(\frac{2}{7} u^{7/2} - \frac{24}{5} u^{5/2} + 36 \cdot \frac{2}{3} u^{3/2} \right) \Big|_6^{11}$

anti derivative

$$\approx \frac{1}{125} \cdot (210.5583 - 80.6232) \approx \frac{129.9351}{125}$$

$$\approx \boxed{1.03948}$$

Example 6 (similar to Example 5)

Consider the definite integral $\int_1^2 x^5 \sqrt{x^3+2} dx$.

- Then the most appropriate substitution to simplify this integral is $u = \underline{\hspace{2cm}}$. Then $dx = f(x)du$ where $f(x) = \underline{\hspace{2cm}}$.
- After making the substitution and simplifying we obtain the integral

$$\int_a^b g(u) du$$

where $g(u) = \underline{\hspace{2cm}}$, $a = \underline{\hspace{1cm}}$ and $b = \underline{\hspace{1cm}}$.

- This definite integral has value = $\underline{\hspace{2cm}}$.

Example 7 (Online Homework # 10)

Consider the indefinite integral $\int \frac{1}{3x+7\sqrt{x}} dx$.

- Then the most appropriate substitution to simplify this integral is $u = \underline{\hspace{2cm}}$. Then $dx = f(x)du$ where $f(x) = \underline{\hspace{2cm}}$.
- After making the substitution and simplifying we obtain the integral $\int g(u) du$ where $g(u) = \underline{\hspace{2cm}}$.
- This last integral is: $= \underline{\hspace{2cm}} + C$.
(Leave out constant of integration from your answer.)
- After substituting back for u we obtain the following final form of the answer: $= \underline{\hspace{2cm}} + C$.

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Lecture 3

This example $\int_1^2 x^5 \sqrt{x^3+2} dx$ is similar to Example 5

Set $\boxed{u = x^3+2}$; so that $\frac{du}{dx} = 3x^2$ or $\boxed{x^2 dx = \frac{1}{3} du}$

Thus we can rewrite the integral as:

$$\int_1^2 x^5 \sqrt{x^3+2} dx = \int_1^2 \underbrace{x^3}_{u-2} \cdot \underbrace{\sqrt{x^3+2}}_{\sqrt{u}} \cdot \underbrace{x^2 dx}_{\frac{1}{3} du}$$

Thus $\boxed{\int_3^{10} \frac{1}{3} (u-2) \sqrt{u} du}$ as $x=1 \rightsquigarrow u=3$
 $x=2 \rightsquigarrow u=10$

$$= \int_3^{10} \left(\frac{1}{3} u^{3/2} - \frac{2}{3} u^{1/2} \right) du = \left[\frac{1}{3} \cdot \frac{2}{5} u^{5/2} - \frac{2}{3} \cdot \frac{2}{3} u^{3/2} \right]_3^{10}$$

$$\cong 28.109 - (-0.231) \cong \underline{\underline{28.34}}$$

Consider the integral $\int \frac{1}{3x+7\sqrt{x}} dx$.

The idea to solve this integral is to notice that $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$. How does this help? Rewrite the integral as follows:

$$\int \frac{1}{3x+7\sqrt{x}} dx = \int \frac{1}{\sqrt{x}(3\sqrt{x}+7)} dx$$

Now set $\boxed{u = 3\sqrt{x}+7}$, so that $\frac{du}{dx} = 3 \cdot \frac{1}{2\sqrt{x}}$.

Thus $\boxed{dx = \frac{2}{3}\sqrt{x} du}$. After making the

substitution we obtain: $\int \underbrace{\frac{2}{3} \cdot \frac{1}{u}}_{g(u)} du$.

$$= \frac{2}{3} \ln|u| + C = \boxed{\frac{2}{3} \ln|3\sqrt{x}+7| + C}$$