MA 138 – Calculus 2 with Life Science Applications
Integration by Parts
(Section 7.2)

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Section 7.2: Integration by Parts

We saw that **integration by parts is the product rule in integral form.**

We also recall the following **general formula**:

Rule for Integration by Parts

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx;$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Example 1

A particle that moves along a straight line has velocity

$$v(t) = t^2 e^{-2t}$$

meters per second after t seconds.
How many meters will it travel during the first t seconds?

Using the notion of cummulative change

$$\underbrace{s(t) - s(0)}_{\text{distance traveled}} = \int_0^t \frac{ds}{du} \cdot du = \int_0^t v(u) \cdot du$$

$$= \int_0^t u^2 \cdot e^{-2u} \cdot du \quad = \quad \text{set} \quad w = -2u \quad \frac{dw}{du} = -2$$

$$\boxed{u = -\tfrac{1}{2} w} \qquad \boxed{du = -\tfrac{1}{2}\, dw}$$

Substitute and change the integration limits:

$$= \int_0^{-2t} \left(-\tfrac{1}{2} w\right)^2 \cdot e^w \cdot \left(-\tfrac{1}{2}\, dw\right) = -\int_0^{-2t} \frac{1}{8} w^2 e^w\, dw =$$

$$= \int_{-2t}^{0} \frac{1}{8} w^2 e^w\, dw \quad = \text{if you prefer relabel} \quad w \longleftrightarrow x$$

$$\therefore\ s(t)-s(0) = \int_{-2t}^{0} \underbrace{\tfrac{1}{8}x^2}_{f}\,\underbrace{e^x}_{g'}\,dx = \text{integration by parts}$$

$$= \tfrac{1}{8}x^2\cdot e^x\Big]_{-2t}^{0} - \int_{-2t}^{0}\tfrac{1}{8}\,2x\cdot e^x\,dx$$

$$= \left(0-\tfrac{1}{8}(-2t)^2e^{-2t}\right) - \int_{-2t}^{0}\tfrac{1}{4}\,x\,e^x\,dx$$

integrate by parts again

$$= -\tfrac{1}{2}t^2e^{-2t} - \left\{\left[\tfrac{1}{4}x\,e^x\right]_{-2t}^{0} - \int_{-2t}^{0}\tfrac{1}{4}e^x\,dx\right\}$$

$$= -\tfrac{1}{2}t^2e^{-2t} - \left\{\left(0-\tfrac{1}{4}(-2t)e^{-2t}\right) - \left[\tfrac{1}{4}e^x\right]_{-2t}^{0}\right\}$$

$$= -\tfrac{1}{2}t^2e^{-2t} - \tfrac{1}{2}te^{-2t} + \left[\tfrac{1}{4}e^0 - \tfrac{1}{4}e^{-2t}\right]$$

$$\boxed{= \tfrac{1}{4} - \left(\tfrac{1}{2}t^2+\tfrac{1}{2}t+\tfrac{1}{4}\right)e^{-2t}}$$

Example 2 (Online Homework # 9)

Suppose that $f(1)=4$, $f(4)=6$, $f'(1)=-5$, $f'(4)=-5$, and f'' is continuous. Find the value of

$$\int_1^4 x\,f''(x)\,dx.$$

$$\int_1^4 x\,f''(x)\,dx = ?$$

$f(1)=4$

$f(4)=6$

$f'(1)=-5$

$f'(4)=-5$

f'' continuous

integrate by parts:

$$\int_1^4 x\,\underbrace{(f')'}_{f''}\,dx = \left[x\,f'\right]_1^4 - \int_1^4 \underbrace{1\cdot f'\,dx}_{\text{F.T.C.}} =$$

$$= \left[x\,f'(x)\right]_1^4 - \left[f(x)\right]_1^4 =$$

$$= \left[4\cdot f'(4) - 1\cdot f'(1)\right] - \left[f(4)-f(1)\right]$$

$$= 4(-5) - (-5) - (6-4) = \boxed{-17}$$

Example 3 (Problem # 8, Section 7.2, page 372)

Evaluate the indefinite integral: $\displaystyle\int 3xe^{-x/2}\,dx.$

$$\int 3x\, e^{-x/2}\, dx = \text{set } \boxed{u = -\frac{x}{2}} \qquad \frac{du}{dx} = -\frac{1}{2}$$

$$\therefore \boxed{x = -2u}, \quad \boxed{-2\, du = dx}; \quad \text{substitute}$$

$$= \int -6u\, e^{u}(-2\,du) = \int \underbrace{12u}_{f}\, \underbrace{e^{u}}_{g'}\, du \underset{\text{by parts}}{=}$$

$$= \underbrace{12u}_{f} \cdot \underbrace{e^{u}}_{g} - \int \underbrace{12}_{f'}\, \underbrace{e^{u}}_{g}\, du = 12u\, e^{u} - 12e^{u} + C$$

$$= \left(12\left(-\frac{x}{2}\right) - 12\right) e^{-x/2} + C = \boxed{-6(x+2)e^{-x/2} + C}$$

Example 4 (Online Homework # 6)

Find the integral: $\displaystyle\int_0^1 x^2 \sqrt[4]{e^x}\, dx.$

$$\int_0^1 x^2 \sqrt[4]{e^x}\, dx = \int_0^1 x^2 (e^x)^{1/4}\, dx = \int_0^1 x^2 e^{x/4}\, dx$$

Use the substitution $\boxed{u = \frac{x}{4}}$ so $\boxed{x = 4u}$

and $\frac{du}{dx} = 1/4$ so $\boxed{4\,du = dx}$. Change also the limits of integration

$$\therefore \int_0^1 x^2 \sqrt[4]{e^x}\, dx = \int_0^{1/4} (4u)^2 \cdot e^u \cdot (4\,du) = \boxed{\int_0^{1/4} 64\, u^2 \cdot e^u\, du}$$

$$\int \underbrace{64u^2}_{f} \cdot \underbrace{e^u}_{g'} \cdot du \underset{\text{by parts}}{=} \underbrace{64u^2}_{f} \cdot \underbrace{e^u}_{g} - \int \underbrace{128u}_{f'} \cdot \underbrace{e^u}_{g}\, du =$$

$$\underset{\text{by parts again}}{=} 64u^2 e^u - \left[128u\, e^u - \int 128 \cdot e^u\, du\right]$$

In conclusion:

$$\int_0^1 x^2 \sqrt[4]{e^x}\, dx = \int_0^{1/4} 64\, u^2 \cdot e^u\, du = \left[\begin{matrix} 64u^2 e^u - 128u e^u \\ + 128 e^u \end{matrix}\right]_0^{1/4}$$

$$= \left[64(u^2 - 2u + 2)e^u\right]_0^{1/4}$$

$$= 64\left(\frac{1}{16} - \frac{1}{2} + 2\right)e^{1/4} - 64(2) \cdot e^0 =$$

$$= 64\left(\frac{1 - 8 + 32}{16}\right)e^{1/4} - 128 = \underline{100e^{1/4} - 128}$$

$$\approx \underline{0.4025}$$

Example 5 (Problem # 35, Section 7.2, page 373)

Evaluate the indefinite integral: $\displaystyle\int \frac{1}{x} \ln x \, dx.$

(1) Our textbook suggests to compute $\int \frac{1}{x} \ln x \, dx$ using integration by parts.

$$\int \underbrace{\frac{1}{x}}_{g'} \cdot \underbrace{\ln x}_{f} \, dx = \underbrace{(\ln x)}_{g} \cdot \underbrace{\ln(x)}_{f} - \int \underbrace{(\ln x)}_{g} \cdot \underbrace{\frac{1}{x}}_{f'} \, dx$$

move this to the right-hand side

$$\therefore \quad 2\int \frac{1}{x} \cdot \ln x \, dx = (\ln x)^2 + C$$

$$\therefore \quad \boxed{\int \frac{1}{x} \ln x \, dx = \frac{1}{2} (\ln x)^2 + \tilde{C}}$$

(2) Using the substitution $\boxed{u = \ln x}$ so $\boxed{du = \frac{1}{x} dx}$

we also get: $\displaystyle\int \frac{1}{x} \cdot \ln x \, dx = \int u \cdot du = \frac{1}{2} u^2 + \tilde{C} = \boxed{\frac{1}{2} (\ln x)^2 + \tilde{C}}$

Example 6 (Problem # 48, Section 7.2, page 373)

Evaluate the definite integral: $\displaystyle\int_0^1 x^3 \ln(x^2+1) \, dx.$

$\displaystyle\int_0^1 x^3 \cdot \ln(x^2+1) \, dx =$ use first the substitution $\boxed{u = x^2+1}$

so that $\frac{du}{dx} = 2x$ or $\boxed{\frac{1}{2} du = x \, dx}$

and observe $\boxed{x^2 = u-1}$

Substitute and change the limits of integration

$$\int_0^1 x^2 \cdot \ln(x^2+1) \cdot x \, dx = \int_1^2 (u-1) \cdot \ln(u) \cdot \frac{1}{2} du =$$

$$= \int_1^2 \underbrace{\left(\frac{1}{2}u - \frac{1}{2}\right)}_{g'} \cdot \underbrace{\ln u}_{f} \, du \underset{\text{by parts}}{=} \left[\left(\frac{1}{4}u^2 - \frac{1}{2}u\right) \ln u\right]_1^2 - \int_1^2 \left(\frac{1}{4}u^2 - \frac{1}{2}u\right) \frac{1}{u} du$$

$$= \left(\frac{1}{4}u^2 - \frac{1}{2}u\right) \ln u \Big]_1^2 - \int_1^2 \left(\frac{1}{4}u - \frac{1}{2}\right) du =$$

$$= \left[\underbrace{\left(\frac{1}{4} \cdot 2^2 - \cancel{\frac{1}{2} \cdot 2}\right)}_{0} \cdot \ln(2) - \left(\frac{1}{4}(1)^2 - \frac{1}{2}(1)\right) \cdot \underbrace{\cancel{\ln(1)}}_{0}\right] - \left[\frac{1}{8}u^2 - \frac{1}{2}u\right]_1^2$$

$$= -\left[\left(\frac{1}{8} \cdot 4 - 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right] = \boxed{\frac{1}{8}} \approx 0.125$$

Useful aside: Trigonometric addition formulas

- We also used the double angle formulas

$$\begin{aligned}\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha\end{aligned} \quad \text{and} \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

to compute $\int \cos^2 x\,dx$ and $\int \sin x \cos x\,dx$.

- Is there a 'simple' way to remember formulas of this kind?

- **Euler's formula** establishes the fundamental relationship between the trigonometric functions and the complex exponential function. It states that, for any real number x,

$$\boxed{e^{ix} = \cos x + i\sin x,}$$

where i is the imaginary unit ($i^2 = -1$).

- For any α and β, using Euler's formula, we have

$$\begin{aligned}\cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{i(\alpha+\beta)} \\ &= e^{i\alpha}\cdot e^{i\beta} \\ &= (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta + i^2\sin\alpha\sin\beta) \\ &\qquad + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).\end{aligned}$$

- Thus, by comparing the terms, we obtain

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

- Thus, by setting $\alpha = \beta$, we obtain

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad \text{and} \quad \sin(2\alpha) = 2\sin\alpha\cos\beta.$$