MA 138 – Calculus 2 with Life Science Applications
Rational Functions and Partial Fractions
(Section 7.3)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Example 1

Evaluate the following indefinite integrals

- $\displaystyle\int \frac{5}{(3x+2)^4}\,dx;$
- $\displaystyle\int \frac{2x-2}{(x^2-2x+5)^3}\,dx.$

(1) $\displaystyle\int \frac{5}{(3x+2)^4}\,dx$ set $\boxed{u = 3x+2}$ so that $\frac{du}{dx} = 3$

$\boxed{dx = \frac{1}{3}\,du}$

hence if we substitute back

$$= \int \frac{5}{u^4}\cdot\frac{1}{3}\,du = \int \frac{5}{3}\,u^{-4}\,du = \frac{5}{3}\cdot\frac{1}{-3}\,u^{-3} + C$$

$$= -\frac{5}{9}\cdot\frac{1}{u^3} + C = \boxed{-\frac{5}{9}\cdot\frac{1}{(3x+2)^3} + C}$$

(2) $\displaystyle\int \frac{2x-2}{(x^2-2x+5)^3}\,dx$ set $\boxed{u = x^2-2x+5}$ so

$\frac{du}{dx} = 2x-2$ $\boxed{(2x-2)\,dx = du}$

hence

$$= \int \frac{du}{u^3} = \int u^{-3}\,du = -\frac{1}{2}\,u^{-2} + C = -\frac{1}{2u^2} + C$$

$$= \boxed{-\frac{1}{2(x^2-2x+5)^2} + C}$$

Section 7.3: Rational Functions and Partial Fractions

- A rational function f is the quotient of two polynomials. That is,
$$f(x) = \frac{P(x)}{Q(x)}$$
where $P(x)$ and $Q(x)$ are polynomials.
- To integrate such a function, we write $f(x)$ as a sum of a polynomial and simpler rational functions (=partial-fraction decomposition).
- These simpler rational functions, which can be integrated with the methods we have learned, are of the form
$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^n}$$
where $A, B, C, a, b,$ and c are constants and n is a positive integer.
- In this form, the quadratic polynomial ax^2+bx+c can no longer be factored into a product of two linear functions with real coefficients.

Proper Rational Functions

- The rational function $f(x) = P(x)/Q(x)$ is said to be **proper** if the degree of the polynomial in the numerator, $P(x)$, is strictly less than the degree of the polynomial in the denominator, $Q(x)$,

$$f(x) = \frac{P(x)}{Q(x)} \quad \textbf{proper} \quad \Longleftrightarrow \quad \deg P(x) < \deg Q(x).$$

- Which of the following three rational functions

$$f_1(x) = \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} \qquad f_2(x) = \frac{x}{x+2} \qquad f_3(x) = \frac{2x-3}{x^2+x}$$

is proper? Only $f_3(x)$ is proper.

- The first step in the partial-fraction decomposition procedure is to use the long division algorithm to write $f(x)$ as a sum of a polynomial and a **proper** rational function.

Algebra Review

Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials.*

Long Division Algorithm

If $A(x)$ and $B(x)$ are polynomials, with $B(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree strictly less than the degree of $B(x)$, such that

$$A(x) = Q(x) \cdot B(x) + R(x)$$

The polynomials $A(x)$ and $B(x)$ are called the **dividend** and **divisor**, respectively; $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

Example 2

Divide the polynomial

$$A(x) = 2x^2 - x - 4 \quad \text{by} \quad B(x) = x - 3.$$

$$\begin{array}{r|l} & 2x \\ \hline x - 3 & 2x^2 - x - 4 \\ & 2x^2 - 6x \\ \hline & \quad + 5x - 4 \end{array}$$

(Complete the above table and check your work!)

$$\begin{array}{r|ll} & 2x + 5 & \\ \hline x - 3 & 2x^2 - x - 4 & \\ & 2x^2 - 6x & \text{subtract} \\ \hline & 5x - 4 & \\ & 5x - 15 & \text{subtract} \\ \hline & 11 & \leftarrow \text{remainder} \end{array}$$

This means that

$$\boxed{2x^2 - x - 4 = (x-3)\cdot(2x+5) + 11}$$

check: $(2x^2 + 5x - 6x - 15) + 11$
$= 2x^2 - x - 4$ ✓

- **Synthetic division** is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$, where c is a number. In synthetic division we write only the essential part of the long division table.
- In synthetic division we abbreviate the polynomial $A(x)$ by writing only its coefficients. Moreover, instead of $B(x) = x - c$, we simply write 'c.' Writing c instead of $-c$ allows us to add instead of subtract!

Example 2 (revisited): Divide

$A(x) = 2x^2 - x - 4$ by $B(x) = x - 3$.

$$\begin{array}{c|ccc} 3 & 2 & -1 & -4 \\ \hline & & & \\ & & 6 & 15 \\ \hline & \boxed{2} & 5 & 11 \end{array}$$

We obtain $Q(x) = 2x + 5$ and $R(x) = 11$. That is,

$$2x^2 - x - 4 = (2x+5)(x-3) + 11.$$

Example 3 (Online Homework # 3)

Use the Long Division Algorithm to write $f(x)$ as a sum of a polynomial and a proper rational function

$$f(x) = \frac{x^3}{x^2 + 4x + 3}.$$

$$\begin{array}{r|l} & x \quad -4 \\ \hline x^2+4x+3 & x^3 \\ & x^3 + 4x^2 + 3x \quad \text{subtract} \\ \hline & -4x^2 - 3x \\ & -4x^2 - 16x - 12 \quad \text{subtract} \\ \hline & 13x + 12 \quad \leftarrow \text{remainder} \end{array}$$

This means that

$$\boxed{x^3 = (x-4)(x^2+4x+3) + 13x + 12}$$

Thus $\dfrac{x^3}{x^2+4x+3} = \dfrac{(x-4)(x^2+4x+3) + 13x + 12}{x^2+4x+3}$

Now split the fraction on the right hand side as the sum of 2 fractions:

$$\frac{x^3}{x^2+4x+3} = \underbrace{\frac{(x-4)\cancel{(x^2+4x+3)}}{\cancel{x^2+4x+3}}}_{\text{simplify}} + \frac{13x+12}{x^2+4x+3}$$

$$= x - 4 + \frac{13x+12}{x^2+4x+3}$$

this is a proper rational fraction

Partial Fraction Decomposition (linear factors)

Case of Distinct Linear Factors

$Q(x)$ is a product of m distinct linear factors. $Q(x)$ is thus of the form

$$Q(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the m distinct roots of $Q(x)$.
The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_m}{x-\alpha_m}\right]$$

We will see in the next examples how the constants $A_1, A_2, \ldots, A_m$ are determined.

Example 3 (cont.d)

Evaluate the indefinite integral: $\displaystyle\int \frac{x^3}{x^2+4x+3}\,dx.$

Note: from the calculations carried out in the first part of the example, we know that our problem reduces to

$$\int (x-4)\,dx + \int \frac{13x+12}{(x+3)(x+1)}\,dx.$$

We have shown before that

$$\frac{x^3}{x^2+4x+3} = x-4 + \frac{13x+12}{x^2+4x+3}$$

Thus :

$$\int \frac{x^3}{x^2+4x+3}\,dx = \underbrace{\int (x-4)\,dx}_{\text{easy}} + \underbrace{\int \frac{13x+12}{x^2+4x+3}\,dx}_{\text{let's look at this}}$$

$$= \tfrac{1}{2}x^2 - 4x + C$$

Notice :

$$\frac{13x+12}{x^2+4x+3} = \frac{13x+12}{(x+1)(x+3)} \underset{\text{want}}{=} \frac{A}{x+3} + \frac{B}{x+1}$$

for some constants A and B

$$\frac{13x+12}{(x+1)(x+3)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1) + B(x+3)}{(x+3)(x+1)}$$

give common denominator

$$= \frac{Ax + A + Bx + 3B}{(x+3)(x+1)} = \frac{(A+B)x + (A+3B)}{(x+1)(x+3)}$$

This means that

$$13x + 12 = (A+B)x + (A+3B)$$

So

$$\begin{cases} A+B = 13 \\ A+3B = 12 \end{cases} \longleftrightarrow \begin{cases} A = 13-B \\ A = 12-3B \end{cases}$$

so $13 - B = 12 - 3B \longleftrightarrow 2B = -1 \longrightarrow \boxed{B = -\tfrac{1}{2}}$

and $A = 13 - B = 13 - (-\tfrac{1}{2}) = \frac{27}{2}$

This means that $\boxed{\dfrac{13x+12}{x^2+4x+3} = \dfrac{27}{2}\cdot\dfrac{1}{x+3} - \dfrac{1}{2}\cdot\dfrac{1}{x+1}}$

Thus

$$\int \frac{13x+12}{x^2+4x+3}\,dx = \frac{27}{2}\int \frac{1}{x+3}\,dx - \frac{1}{2}\int\frac{1}{x+1}\,dx$$

$$= \frac{27}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C$$

Thus:

$$\int \frac{x^3}{x^2+4x+3} = \underbrace{\frac{1}{2}x^2 - 4x} + \underbrace{\frac{27}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C}$$

(Heaviside) cover-up method

We illustrate this method by using the previous example:

$$\frac{13x+12}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$$

$$\rightsquigarrow$$

$$\boxed{A(x+1)+B(x+3) = 13x+12} \qquad (*)$$

Set $x=-1$ in $(*)$. We obtain

$$A\cdot 0 + B\cdot(-1+3) = 13(-1)+12$$
$$\rightsquigarrow$$
$$B\cdot(2) = -1$$
$$\rightsquigarrow$$
$$B = -1/2$$

Set $x=-3$ in $(*)$. We obtain

$$A\cdot(-3+1)+0 = 13(-3)+12$$
$$\rightsquigarrow$$
$$A\cdot(-2) = -27$$
$$\rightsquigarrow$$
$$A = 27/2$$

Example 4 (Online Homework # 8)

Find the integral: $\displaystyle\int_2^5 \frac{2}{x^2-1}\,dx.$

Consider the fraction $\dfrac{2}{x^2-1} = \dfrac{2}{(x-1)(x+1)}$.

we want to decompose it as:

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \text{ = give common denominator}$$

$$= \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

Thus $2 = A(x+1) + B(x-1)$

evaluate at $\boxed{x=1}$: $\quad 2 = 2\cdot A + B\cdot 0 \quad \therefore \boxed{A=1}$

evaluate at $\boxed{x=-1}$: $\quad 2 = A\cdot(0) + B(-2) \quad \therefore \boxed{B=-1}$

Thus: $\boxed{\dfrac{2}{x^2-1} = \dfrac{1}{x-1} - \dfrac{1}{x+1}}$

Thus:
$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+1}\,dx$$
$$= \ln|x-1| - \ln|x+1| + C$$
$$= \ln\left|\frac{x-1}{x+1}\right| + C$$

Finally:
$$\int_2^5 \frac{2}{x^2-1}\,dx = \ln\left|\frac{x-1}{x+1}\right|\Bigg]_2^5 =$$
$$= \ln\left(4/6\right) - \ln\left(1/3\right)$$
$$= \ln\left(4/6 \cdot 3\right) = \underline{\underline{\ln(2)}} \approx \underline{0.6931}$$

Example 5 (Online Homework # 6)

Evaluate the indefinite integral: $\displaystyle\int \frac{1}{x(x+1)}\,dx.$

We need to decompose the fraction $\frac{1}{x(x+1)}$ as $\frac{A}{x} + \frac{B}{x+1}$. Thus:
$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+B\cdot x}{x(x+1)}$$

Thus $1 = A(x+1) + Bx$

- evaluate at $\boxed{x=0}$

$1 = A\cdot 1 + B\cdot 0$

$\therefore \boxed{A=1}$

- evaluate at $\boxed{x=-1}$

$1 = A\cdot 0 + B(-1)$

$\therefore \boxed{B=-1}$

$(A+B)x + A = 1$ implies
$$\begin{cases} A+B=0 \\ A=1 \end{cases}$$
$\therefore \boxed{A=1}\ B=-1$

No matter which method we choose, we obtain
$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

Thus
$$\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$
$$= \ln|x| - \ln|x+1| + C$$
$$= \underline{\ln\left|\frac{x}{x+1}\right| + C}$$