

MA 138 – Calculus 2 with Life Science Applications
 Rational Functions and Partial Fractions
 (Section 7.3)

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Example 1 (Online Homework # 7)

Evaluate the indefinite integral: $\int \frac{3}{(x+a)(x+b)} dx.$

We need to consider 2 cases:

$a = b$ the integral becomes $\int \frac{3}{(x+a)^2} dx$

so by setting $u = x+a$, $\frac{du}{dx} = 1$ so $du = dx$

Thus $= \int \frac{3}{u^2} du = \int 3u^{-2} du = -3u^{-1} + C$
 $= -\frac{3}{u} + C = \boxed{-\frac{3}{x+a} + C}$

$a \neq b$ We need to write $\frac{3}{(x+a)(x+b)}$ as $\frac{A}{x+a} + \frac{B}{x+b}$

for some A, B constants.

$$\frac{3}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} = \frac{A(x+b) + B(x+a)}{(x+a)(x+b)}$$

Thus the numerators must be the same:

$$3 = A(x+b) + B(x+a)$$

$$= (A+B)x + (Ab+Ba)$$

$\therefore A+B=0$ and $Ab+Ba=3$

$\hookrightarrow A=-B$ and $-Bb+Ba=3$

$\therefore \boxed{B = \frac{3}{a-b}}$ and $\boxed{A = -B = \frac{-3}{a-b}}$

Thus $\frac{3}{(x-a)(x+b)} = \frac{-3/a-b}{x+a} + \frac{3/a-b}{x+b}$

$$= \frac{3}{a-b} \left[\frac{1}{x+b} - \frac{1}{x+a} \right]$$

$\therefore \int \frac{3}{(x+a)(x+b)} dx = \frac{3}{a-b} \left[\int \frac{1}{x+b} dx - \int \frac{1}{x+a} dx \right]$

$$\therefore \int \frac{3}{(x+a)(x+b)} dx = \frac{3}{a-b} \left[\ln|x+b| - \ln|x+a| \right] + C$$

$$= \boxed{\frac{3}{a-b} \cdot \ln \left| \frac{x+b}{x+a} \right| + C}$$

$$\left[= \frac{3}{b-a} \cdot \ln \left| \frac{x+a}{x+b} \right| + C \right]$$

Example 2

Consider the rational function

$$f(x) = \frac{4x^2 - x - 1}{(x+1)^2(x-3)}$$

which has a repeated factor at the denominator.

Try to find constants A and B such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)^2} + \frac{B}{(x-3)}$$

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Lecture 7

Let's try to find A and B such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)^2} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x+1)^2}{(x+1)^2(x-3)}$$

$$= \frac{Ax - 3A + Bx^2 + 2Bx + B}{(x+1)^2(x-3)}$$

$$= \frac{Bx^2 + (A+2B)x + (B-3A)}{(x+1)^2(x-3)}$$

Thus: $B=4$, $A+2B=-1$, $B-3A=-1$

\downarrow
 $A+8=-1 \Rightarrow A=-9$ (But: not $4-3(-9)=-1$)

IMPOSSIBLE!!!

Example 2 (again)

The previous calculation didn't work.

Try now to find constants A , B and C such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-3)}$$

Then evaluate the definite integral

$$\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)} dx.$$

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Lecture 7

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$$

$$= \frac{A(x+1)(x-3) + B(x-3) + C(x+1)^2}{(x+1)^2(x-3)}$$

The denominators are the same:

$$4x^2 - x - 1 = A(x+1)(x-3) + B(x-3) + C(x+1)^2 \quad (*)$$

If we evaluate (*) for $x = -1$ we obtain

$$4(-1)^2 - (-1) - 1 = A \cdot 0 + B(-4) + C \cdot 0$$

$$\therefore -4B = 4 \quad \therefore \boxed{B = -1}$$

If we evaluate (*) at $x = 3$ we obtain

$$4(3)^2 - 3 - 1 = A \cdot 0 + B \cdot 0 + 16C$$

$$\therefore 16C = 32 \quad \therefore \boxed{C = 2}$$

Thus the (Heaviside) cover-up method doesn't produce directly A ! Well substitute in (*) for example $x = 0$ and get:

$$4 \cdot 0^2 - 0 - 1 = A \cdot (1)(-3) + (-1)(0-3) + (2)(0+1)^2$$

$$\text{So: } -1 = -3A + 3 + 2 \quad \leftrightarrow \quad 3A = 6 \quad \therefore \boxed{A = 2}$$

Bottom line:

$$\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)} dx = \int \frac{2}{x+1} dx + \int \frac{-1}{(x+1)^2} dx + \int \frac{2}{x-3} dx$$

$$= 2 \ln|x+1| + \frac{1}{x+1} + 2 \ln|x-3| + C$$

$$= \ln \left[(x+1)^2(x-3)^2 \right] + \frac{1}{x+1} + C$$

using the properties of logarithms

Partial Fraction Decomposition

(repeated linear factors)

Case of Repeated Linear Factors

$Q(x)$ is a product of m distinct linear factors to various powers. $Q(x)$ is thus of the form

$$Q(x) = a(x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \cdots (x - \alpha_m)^{n_m}$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ are the m distinct roots of $Q(x)$ and n_1, n_2, \dots, n_m are positive integers such that $n_1 + n_2 + \cdots + n_m = \deg Q(x)$.

The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a} \left[\sum_{i=1}^m \frac{A_{i,1}}{x - \alpha_i} + \frac{A_{i,2}}{(x - \alpha_i)^2} + \cdots + \frac{A_{i,n_i}}{(x - \alpha_i)^{n_i}} \right]$$

Example 3 (Online Homework # 5)

Evaluate the integral

$$\int \frac{-10x^2}{(x+1)^3} dx.$$

We need to find $A, B,$ and C such that

$$\begin{aligned} \frac{-10x^2}{(x+1)^3} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \\ &= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} \end{aligned}$$

Thus: $-10x^2 = A(x+1)^2 + B(x+1) + C$ (*)

Evaluate (*) at $x=-1$. We get "C"!

$$-10(-1)^2 = A \cdot 0 + B \cdot 0 + C \quad \therefore \boxed{C = -10}$$

If we plug-in in (*) $x=0$ and $x=1$ for example we get:

$$x=0: \quad 0 = A(1)^2 + B(1) - 10$$

$$x=1: \quad -10 = A(2)^2 + B(2) - 10$$

$$\text{So: } \begin{cases} A+B=10 \\ 4A+2B=0 \end{cases} \rightarrow \begin{cases} B=-2A \\ A+(-2A)=10 \end{cases}$$

$$\therefore \boxed{A = -10} \quad \text{and} \quad \boxed{B = 20}$$

$$\text{Thus: } \frac{-10x^2}{(x+1)^3} = \frac{-10}{x+1} + \frac{20}{(x+1)^2} + \frac{-10}{(x+1)^3}$$

$$\begin{aligned} \text{and } \int \frac{-10x^2}{(x+1)^3} dx &= -10 \int \frac{1}{x+1} dx + 20 \int \frac{1}{(x+1)^2} - 10 \int \frac{1}{(x+1)^3} dx \\ &= -10 \ln|x+1| + 20 \left[-1 \cdot (x+1)^{-1} \right] - 10 \left[-\frac{1}{2} (x+1)^{-2} \right] + C \end{aligned}$$

$$\begin{aligned} &\boxed{-10 \ln|x+1| + \frac{-20}{x+1} + \frac{5}{(x+1)^2} + C} \\ &= -10 \ln|x+1| + \frac{-20x-15}{(x+1)^2} + C \end{aligned}$$

$$\int \frac{-10x^2}{(x+1)^3} dx \quad \text{you can also } u=x+1 \quad du=dx$$

$$\Rightarrow x=u-1$$

$$= \int \frac{-10(u-1)^2}{u^3} du = \int \frac{-10(u^2-2u+1)}{u^3} du$$

$$= \int \left[-10 \frac{u^2}{u^3} + 20 \frac{u}{u^3} - 10 \frac{1}{u^3} \right] du =$$

$$= \int \left[-10 \frac{1}{u} + 20 \frac{1}{u^2} - 10 \frac{1}{u^3} \right] du =$$

$$= -10 \ln|u| - 20 \frac{1}{u} + 5 \frac{1}{u^2} + C$$

$$= \boxed{-10 \ln|x+1| - 20 \frac{1}{x+1} + 5 \frac{1}{(x+1)^2} + C}$$

Example 4

Evaluate the integral

$$\int \frac{1}{x^2(x-1)^2} dx.$$

We want to write $\frac{1}{x^2(x-1)^2}$ as the following sum of fractions

$$\frac{1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

for some constants $A, B, C,$ and D . Give common denom -

$$\begin{aligned} &= \frac{Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2}{x^2(x-1)^2} \\ &= \frac{A(x^3 - 2x^2 + x) + B(x^2 - 2x + 1) + C(x^3 - x^2) + Dx^2}{x^2(x-1)^2} \\ &= \frac{x^3[A+C] + x^2[-2A+B-C+D] + x[A-2B] + D}{x^2(x-1)^2} \end{aligned}$$

Equate the coefficients of the denominators:

$$A+C=0; \quad -2A+B-C+D=0; \quad A-2B=0; \quad B=1$$

Thus $B=1; \quad A=2B=2; \quad C=-A=-2$.

Finally from $-2A+B-C+D=0$ we get

$$-4+1-(-2)+D=0 \quad \text{so } D=1$$

Hence:

$$\frac{1}{x^2(x-1)^2} = \frac{2}{x} + \frac{1}{x^2} + \frac{-2}{x-1} + \frac{1}{(x-1)^2} \quad \text{using this, we get}$$

$$\int \frac{1}{x^2(x-1)^2} dx = \boxed{2 \ln|x| - \frac{1}{x} - 2 \ln|x-1| - \frac{1}{(x-1)} + C}$$

$$\left[= \ln \left[\frac{x^2}{(x-1)^2} \right] + \frac{1-2x}{x(x-1)} + C \right]$$

after simplifying and using the properties of logarithms

Example 5 (Online Homework #11)

If $f(x)$ is a quadratic function such that $f(0) = 1$ and $\int \frac{f(x)}{x^2(x+1)^3} dx$ is a rational function, find the value of $f'(0)$.

$f(x)$ is a quadratic function; so $f(x) = ax^2 + bx + c$.

We know that $f(0) = 1$ so $f(0) = a \cdot 0^2 + b \cdot 0 + c = \boxed{c=1}$

Thus $f(x) = ax^2 + bx + 1$. We are asked to compute

$f'(0)$. Now $f'(x) = 2ax + b$. Hence we are asked to compute $f'(0) = 2a \cdot 0 + b = \boxed{b}$

We are asking that

$$\int \frac{f(x)}{x^2(x+1)^3} dx = \int \frac{ax^2 + bx + 1}{x^2(x+1)^3} dx \text{ is a rational function.}$$

Think about the partial fraction decomposition

$$\frac{ax^2 + bx + 1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

If A and C were to be $\neq 0$ those 2 fractions would give rise to logarithmic terms. Thus we need a partial fraction decomposition of the form:

$$\begin{aligned} \frac{ax^2+bx+1}{x^2(x+1)^3} &= \frac{B}{x^2} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3} \\ &= \frac{B(x+1)^3 + Dx^2(x+1) + Ex^2}{x^2(x+1)^3} \\ &= \frac{B(x^3+3x^2+3x+1) + D(x^3+x^2) + Ex^2}{x^2(x+1)^3} \\ &= \frac{x^3[B+D] + x^2[3B+D+E] + 3Bx + B}{x^2(x+1)^3} \end{aligned}$$

Thus the coefficients of the numerators must be the same:

$$B+D=0 \quad 3B+D+E=a \quad 3B=b \quad B=1$$

Thus $B=1$ so $b=3B=3$ ←←

Note that $D=-B=-1$ so that

$$E = a - 3B - D = a - 3 - (-1) = a - 2$$

Hence

$$\frac{ax^2+3x+1}{x^2(x+1)^3} = \frac{1}{x^2} - \frac{1}{(x+1)^2} + \frac{a-2}{(x+1)^3}$$

Partial Fraction Decomposition (irreducible quadratic factors)

Irreducible quadratic factors in the denominator of a proper rational function are dealt with in the partial-fraction decomposition as follows:

Case of Irreducible Quadratic Factors

If the irreducible quadratic factor $ax^2 + bx + c$ is contained n times in the factorization of the denominator of a proper rational function, then the partial-fraction decomposition contains terms of the form

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \dots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Example 6 (Example 11, Section 7.3, page 384)

Write the partial fraction decomposition of

$$f(x) = \frac{2x^3 - x^2 + 2x - 2}{(x^2 + 1)(x^2 + 2)}$$

$$\frac{2x^3 - x^2 + 2x - 2}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

for some A, B, C, and D.

$$\begin{aligned} &= \frac{(Ax+B)(x^2+2) + (Cx+D)(x^2+1)}{(x^2+1)(x^2+2)} \\ &= \frac{Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 + Cx + Dx^2 + D}{(x^2+1)(x^2+2)} \\ &= \frac{x^3[A+C] + x^2[B+D] + x[2A+C] + 2B+D}{(x^2+1)(x^2+2)} \end{aligned}$$

So that, if we equate the coefficients of the numerators

$$\boxed{A+C=2}$$

$$\boxed{B+D=-1}$$

$$\boxed{2A+C=2}$$

$$\boxed{2B+D=-2}$$

So we need to solve

$$\begin{cases} A+C=2 \\ 2A+C=2 \end{cases}$$

$$\begin{cases} B+D=-1 \\ 2B+D=-2 \end{cases}$$

These 2 systems give

$$A=0 \quad C=2$$

$$B=-1 \quad D=0$$

Hence

$$\boxed{\frac{2x^3 - x^2 + 2x - 2}{(x^2+1)(x^2+2)} = \frac{-1}{x^2+1} + \frac{2x}{x^2+2}}$$

$$\left[\int \frac{2x^3 - x^2 + 2x - 2}{(x^2+1)(x^2+2)} = \ln(x^2+2) - \tan^{-1}x + C \right]$$