MA 138 – Calculus 2 with Life Science Applications
Linear Systems
(Section 9.1)

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Arbitrary Systems of Linear Equations

A system of m equations in n variables can be written in the form

$$\left\{\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}\right.$$

- The variables are now $x_1, x_2, \ldots, x_n$.
- The coefficients a_{ij} on the left-hand side have two subscripts. The first subscript (that is, 'i') indicates the equation, and the second subscript (that is 'j') indicates to which variable a_{ij} corresponds to.
- Double subscripts are a convenient way of labeling the coefficients.
- The subscripts on the b_i's on the right-hand side indicate the equation.

The Gaussian Elimination Method

We transform the given system of linear equations into an equivalent one ($\equiv$ the new system has the same solutions as the old one) **in upper triangular form**.

To do so, we will use the following three basic operations:

1. multiplying an equation by a nonzero constant
2. adding one equation to another
3. rearranging the order of the equations

This method is also called **Gaussian elimination method**.

As seen before, the general linear system may have

- exactly one solution
- no solution ($\equiv$ we say that the system is inconsistent)
- infinitely many solutions

As seen before, the three basic operations in the Gaussian elimination method make changes only to the coefficients of the variables. Thus we will work on the **augmented matrix**

$$\begin{array}{c} \\ \text{1st row} \longrightarrow \\ \text{2nd row} \longrightarrow \\ \\ m\text{th row} \longrightarrow \end{array} \begin{array}{c} \begin{array}{ccccc} \text{1st column} \downarrow & \text{2nd column} \downarrow & & n\text{th column} \downarrow & \end{array} \\ \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \end{array}$$

The entries a_{ij} of the $m \times n$ **matrix** on the left have two subscripts:
The entry a_{ij} is located in the ith row and the jth column.
The $m \times 1$ matrix on the right ($\equiv$ with the b_i's) is called a **column vector**.

Geometric Remarks

'2' we know that systems of linear equations in **two** variables correspond to **intersecting lines in the plane**.

'3' we can visualize that systems of linear equations in **three** variables correspond to **intersecting planes in the space**.

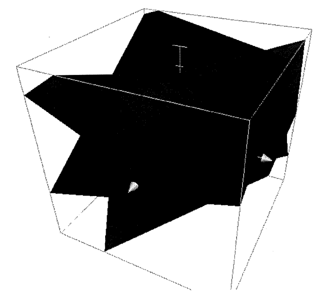
'n' Stretching our imagination, systems of linear equations in $\mathbf{n} \geq \mathbf{4}$ variables correspond **intersecting hyperplanes in the n-dimensional space.**

- Ideally the systems that we would like to encounter have the same number of equations as variables. This need not be the case.
- A system with fewer equations than variables is said to be **underdetermined**. They frequently have infinitely many solutions.
- A system with more equations than variables is said to be **overdetermined**. They frequently are inconsistent.

Example 1

Find the solution of the system of linear equations

$$\begin{cases} 3x_1 + 5x_2 - x_3 = 10 \\ 2x_1 - x_2 + 3x_3 = 9 \\ 4x_1 + 2x_2 - 3x_3 = -1 \end{cases}$$



This is how the configuration of the three planes looks like.

$$\left[\begin{array}{ccc|c} 3 & 5 & -1 & 10 \\ 2 & -1 & 3 & 9 \\ 4 & 2 & -3 & -1 \end{array}\right] \rightsquigarrow \begin{array}{c} \\ \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 5 & -1 & 10 \\ 2 & -1 & 3 & 9 \\ 1 & -3 & -2 & -11 \end{array}\right]$$

augmented matrix

exchange R_3 and R_1

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -11 \\ 2 & -1 & 3 & 9 \\ 3 & 5 & -1 & 10 \end{array}\right] \rightsquigarrow \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -11 \\ 0 & 5 & 7 & 31 \\ 0 & 14 & 5 & 43 \end{array}\right]$$

$$\begin{array}{c} \\ \frac{1}{5}R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -11 \\ 0 & 1 & 7/5 & 31/5 \\ 0 & 14 & 5 & 43 \end{array}\right] \rightsquigarrow \begin{array}{c} \\ \\ R_3 - 14R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -11 \\ 0 & 1 & 7/5 & 31/5 \\ 0 & 0 & -\frac{73}{5} & -219/5 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \\ \\ -\frac{5}{73}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -11 \\ 0 & 1 & 7/5 & 31/5 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightsquigarrow \begin{array}{c} R_1 + 2R_3 \\ R_2 - \frac{7}{5}R_3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} R_1 + 3R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\therefore \quad x_1 = 1 \qquad x_2 = 2 \qquad x_3 = 3$

Example 2 (Problem # 7, Exam 2, Spring '14)

(a) Find the solution(s) for the system of linear equations corresponding to the following augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -7 & 10 \\ 0 & 0 & 0 & -5 \end{array}\right].$$

(b) Find the solution(s) for the system of linear equations corresponding to the following augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

(a) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -7 & 10 \\ 0 & 0 & 0 & -5 \end{array}\right]$ this system has no solutions

It is inconsistent as the last row reads as: $0 = -5$. Impossible.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ this system has infinitely many solutions. It is consistent.

It reads $\begin{cases} x + 4z = 6 \\ y - 5z = -4 \end{cases}$

We can give z any value, say $t \in \mathbb{R}$

Thus $x + 4t = 6$

$y - 5t = -4$

So $\boxed{\begin{array}{ll} x = 6 - 4t & \\ y = -4 + 5t & t \in \mathbb{R} \\ z = t & \end{array}}$

the points of the form

$\{ (6-4t, -4+5t, t) \quad t \in \mathbb{R} \}$

are the infinite solutions. It is a line

Example 3 (Online Homework # 9)

Determine the value of k for which the following system

$$\begin{cases} x + y + 5z = -3 \\ x + 2y - 3z = 0 \\ 3x + 8y + kz = 7 \end{cases}$$

has no solution.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & -3 \\ 1 & 2 & -3 & 0 \\ 3 & 8 & \mathbf{k} & 7 \end{array}\right]$$

We need to find the value k for which the system has no solution. Let's apply the Gaussian elimination process.

$$\begin{array}{c} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 5 & -3 \\ 0 & 1 & -8 & 3 \\ 0 & 5 & k-15 & 16 \end{array}\right] \rightsquigarrow$$

$$\begin{array}{c} \\ \\ R_3 - 5R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 5 & -3 \\ 0 & 1 & -8 & 3 \\ 0 & 0 & k+25 & 1 \end{array}\right]$$

We need $k + 25 = 0$ so that the last equation reads $0z = 1$.

$\therefore$ $\boxed{k = -25}$

Example 4 (Online Homework # 10)

A dietician is planning a meal that supplies certain quantities of vitamin C, calcium and magnesium. Three foods will be used.
The nutrients supplied, measured in milligrams (mg), by one unit of each food and the dietary requirements are given in the table below

Nutrient	Food 1	Food 2	Food 3	Total Required (mg)
Vitamin C	30	60	45	525
Calcium	30	80	65	665
Magnesium	20	55	40	445

The dietician is interested in determining the quantities (in units) x, y and z of Food 1, Food 2, and Food 3, respectively.
Set-up a system of equations for this problem and solve it.

x, y, z are the quantities of Food 1, 2, 3 respectively. We have the following 3 equations:

$$\begin{cases} 30x + 60y + 45z = 525 & \text{requirement of vitamin C} \\ 30x + 80y + 65z = 665 & \text{requirement of Calcium} \\ 20x + 55y + 40z = 445 & \text{requirement of Magnesium} \end{cases}$$

$$\left[\begin{array}{ccc|c} 30 & 60 & 45 & 525 \\ 30 & 80 & 65 & 665 \\ 20 & 55 & 40 & 445 \end{array}\right] \rightsquigarrow \begin{array}{c} \frac{1}{15}R_1 \\ \frac{1}{5}R_2 \\ \frac{1}{5}R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & 3 & 35 \\ 6 & 16 & 13 & 133 \\ 4 & 11 & 8 & 89 \end{array}\right]$$

$$\begin{array}{r}\\ R_2-3R_1\\ R_3-2R_1\end{array}\left[\begin{array}{ccc|c}2&4&3&35\\0&4&4&28\\0&3&2&19\end{array}\right]\rightsquigarrow\begin{array}{r}\frac{1}{2}R_1\\ \frac{1}{4}R_2\\ \\ \end{array}\left[\begin{array}{ccc|c}1&2&3/2&35/2\\0&1&1&7\\0&3&2&19\end{array}\right]$$

$$\begin{array}{r}\\ \\ R_3-3R_2\end{array}\left[\begin{array}{ccc|c}1&2&3/2&35/2\\0&1&1&7\\0&0&-1&-2\end{array}\right]\rightsquigarrow\begin{array}{r}\\ \\ -R_3\end{array}\left[\begin{array}{ccc|c}1&2&3/2&35/2\\0&1&1&7\\0&0&1&2\end{array}\right]$$

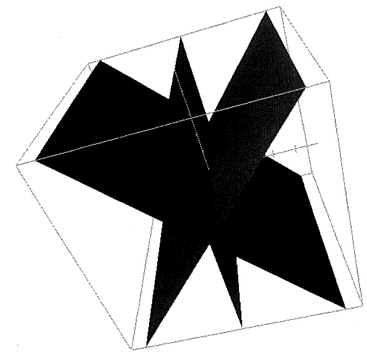
$$\begin{array}{r}R_1-\frac{3}{2}R_3\\ R_2-R_3\\ \\ \end{array}\left[\begin{array}{ccc|c}1&2&0&29/2\\0&1&0&5\\0&0&1&2\end{array}\right]\rightsquigarrow\begin{array}{r}R_1-2R_2\\ \\ \\ \end{array}\left[\begin{array}{ccc|c}1&0&0&9/2\\0&1&0&5\\0&0&1&2\end{array}\right]$$

need 4.5 units of Food 1; 5 units of Food 2; 2 unit of Food 3

Example 5 (Problem # 27, Section 9.1, p. 500)

Find the solution of the system of linear equations

$$\begin{cases} y+x=3\\ z-y=-1\\ x+z=2\end{cases}$$



This is how the configuration of the three planes looks like.

$$\begin{cases} y+x=3\\ z-y=-1\\ x+z=2\end{cases}\Longrightarrow\begin{cases} x+y=3\\ -y+z=-1\\ x+z=2\end{cases}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c}1&1&0&3\\0&-1&1&-1\\1&0&1&2\end{array}\right]\rightsquigarrow\begin{array}{r}\\ \\ R_3-R_1\end{array}\left[\begin{array}{ccc|c}1&1&0&3\\0&-1&1&-1\\0&-1&1&-1\end{array}\right]$$

$$\rightsquigarrow\begin{array}{r}\\ \\ R_3-R_2\end{array}\left[\begin{array}{ccc|c}1&1&0&3\\0&-1&1&-1\\0&0&0&0\end{array}\right]\rightsquigarrow\begin{array}{r}\\ -R_2\\ \\ \end{array}\left[\begin{array}{ccc|c}1&1&0&3\\0&1&-1&1\\0&0&0&0\end{array}\right]$$

$$\rightsquigarrow\begin{array}{r}R_1-R_2\\ \\ \\ \end{array}\left[\begin{array}{ccc|c}1&0&1&2\\0&1&-1&1\\0&0&0&0\end{array}\right]$$

This system is consistent; it has infinitely many solutions. It reads

$$\begin{cases} x+z=2\\ y-z=1\end{cases}$$

If we give z any arbitrary value $t\in\mathbb{R}$ then $x=2-t$ $y=1+t$.

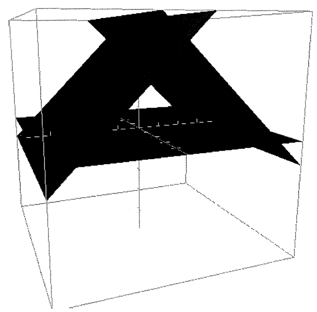
Thus there are a line of solutions:

$$\boxed{\{(2-t,1+t,t)\mid t\in\mathbb{R}\}}$$

Example 6

Find the solution of the system of linear equations

$$\begin{cases} x+y-z=3 \\ x-y+z=3 \\ y-z=1.5 \end{cases}$$



This is how the configuration of the three planes looks like.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1.5 \end{array}\right]$$

If we row reduce we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

which is not consistent
the last row reads: $\boxed{0=1}$
Impossible!
There are no solutions.