MA 138 – Calculus 2 with Life Science Applications
Matrices
(Section 9.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Outline

We already saw that when transforming a system of linear equations into an equivalent one by using the Gaussian Elimination Process, we make changes only to the coefficients of the variables.

For this reason we introduced the notion of an augmented matrix.

Now, we formalize again the notion of a matrix and then we learn various operations that we can perform on matrices.

More precisely, we will focus on
- **basic matrix operations**;
- **matrix multiplication**;
- **inverse of matrices**;
- **application** to solving systems of n linear equations in n variables.

The ... Guiding Light

- *A simple key observation*: To solve $5x = 10$ for x, we just divide both sides by 5 ($\equiv$ multiply both sides by $1/5 = 5^{-1}$). That is,
$$5x = 10 \quad \leadsto \quad 5^{-1} \cdot 5x = 5^{-1} \cdot 10 \quad \leadsto \quad x = 2$$
as $5^{-1} \cdot 5 = 1$ and $5^{-1} \cdot 10 = 2$.
- We will learn how to write a system of n linear equations in n variables in the matrix form $AX = B$.
- To solve $AX = B$, we therefore need an operation that is *analogous to* multiplication by the 'reciprocal' of A. We will define, whenever possible, a matrix A^{-1} that will serve this function (i.e., $A^{-1} \cdot A =$ Identity Matrix). It is called the inverse matrix of A.
- Then, whenever possible, we can write the solution of $AX = B$ as
$$AX = B \quad \leadsto \quad A^{-1} \cdot AX = A^{-1} \cdot B \quad \leadsto \quad X = A^{-1} \cdot B.$$

Matrices

An $m \times n$ matrix A is a rectangular array of numbers with m rows and n columns. We write

$$A = \begin{array}{r c} & \begin{array}{cccc} \text{1st column} \downarrow & \text{2nd column} \downarrow & & m\text{th column} \downarrow \end{array} \\ \begin{array}{r} \text{1st row} \longrightarrow \\ \text{2nd row} \longrightarrow \\ \\ m\text{th row} \longrightarrow \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} = [\, a_{ij} \,]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

- We also use the shorthand notation $A = [a_{ij}]$ whenever the size of the matrix is clear.
- In the entry a_{ij}, i is the row index while j is the column index.

Basic Matrix Operations

Equality of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices, then
$$A = B \iff a_{ij} = b_{ij} \text{ for all } 1 \le i \le m \text{ and } 1 \le j \le n.$$

It says that we can compare matrices of the same size, and they are equal if and only if all their corresponding entries are equal.

Addition of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices, then $C = A + B$ is an $m \times n$ matrix with entries $c_{ij} = a_{ij} + b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

It says that we can add matrices of the same size by adding the corresponding entries of the matrices.

Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar ($\equiv$ number), then cA is an $m \times n$ matrix with entries ca_{ij} for all $1 \le i \le m$ and $1 \le j \le n$.

It says that we can multiply a matrix by any number c ($\equiv$ scalar) by multiplying each entry of the matrix by the number c.

The operation that interchanges rows and columns of a matrix is called transposition.

Transpose of a Matrix

Suppose that $A = [a_{ij}]$ is an $m \times n$ matrix. Then the **transpose** of A, denoted by A^T (or A' in our textbook), is an $n \times m$ matrix with entries
$$a^T_{ij} = a_{ji}$$
for all $1 \le i \le m$ and $1 \le j \le n$.

Example 1

Suppose $A = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a 2×1 matrix ($\equiv$ column vector) then

$A^T = \begin{bmatrix} -2 & 1 \end{bmatrix}$ is a 1×2 matrix ($\equiv$ row vector).

Example 2

Let A and B be the following two 2×3 matrices
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}.$$

Find the following matrices
$$A^T \qquad (A^T)^T \qquad 2A - 3B.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

Then:

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

the first row becomes the first column

the second row becomes the second column

$$(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$$

$$2A - 3B = 2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

$$2A - 3B = \cdots = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} + \begin{bmatrix} -3 & -6 & -9 \\ 3 & 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4-6 & 6-9 \\ 8+3 & 10+0 & 12-12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & -3 \\ 11 & 10 & 0 \end{bmatrix}$$

Matrix Multiplication

Matrix Multiplication ($\equiv$ row and column multiplication)

Suppose that $A = [a_{ij}]$ is an $m \times \ell$ matrix and $B = [b_{ij}]$ is an $\ell \times n$ matrix. Then $C = A \cdot B$ is an $m \times n$ matrix $[c_{ij}]$ with

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{i\ell}b_{\ell j} = \sum_{k=1}^{\ell} a_{ik}b_{kj}$$

for $1 \le i \le m$ and $1 \le j \le n$.

- Note that c_{ij} is the entry in C that is located in the ith row and the jth column. To obtain it, we multiply (and then add) the entries of the ith row of A with the entries of the jth column of B.
- For the product $A \cdot B$ (or AB in short) to be defined, the number of columns in A must equal the number of rows in B.

Pictorially

$$\begin{array}{r} \\ \\ i\text{th row} \longrightarrow \\ \\ \\ \end{array} \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{i\ell} \\ \vdots & \vdots & \vdots & & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{array}{c} \begin{array}{ccccc} & & j\text{th column} \downarrow & & \end{array} \\ \begin{bmatrix} \cdots & \cdots & b_{1j} & \cdots & \cdots \\ \vdots & \vdots & b_{2j} & \vdots & \vdots \\ \vdots & \vdots & b_{3j} & \vdots & \vdots \\ \vdots & & \vdots & & \vdots \\ \cdots & \cdots & b_{\ell j} & \cdots & \cdots \end{bmatrix} \end{array}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{i\ell}b_{\ell j} = \sum_{k=1}^{\ell} a_{ik}b_{kj}$$

Properties of the Operations on Matrices

The following properties assume that all matrices are of appropriate sizes so that all of the indicated matrix operations are defined:

- (**commutativity of addition**) $A + B = B + A$
- (**associativity of addition**) $(A + B) + C = A + (B + C)$
- The matrix with all its entries equal to zero is called the **zero matrix** and is denoted by $\mathbf{0}$. It is such that $A + \mathbf{0} = A$ for any matrix A.
- (**scalar multiplication and addition**)
  $c(A + B) = cA + cB$ and $(c_1 + c_2)A = c_1A + c_2A$
- (**scalar multiplication and matrix multiplication**)
  $c(AB) = (cA)B = A(cB)$

Properties of the Operations on Matrices (cont'd)

- **(associativity of multiplication)** $(AB)C = A(BC)$
- **(distributivity of matrix multiplication over addition)**
  $(A+B)C = AC + BC$ and $A(B+C) = AB + AC$
- $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$
- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(cA)^T = cA^T$

Example 3 (Order Is Important!)

Consider the matrices $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

Find the matrices AB and BA.

$$AB = \underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}}_{2\times 3} =$$ result is a 3×3 matrix

$$= \begin{bmatrix} 3\cdot 1 + 2\cdot 2 & 3\cdot 0 + 2\cdot 1 & 3\cdot 1 + 2\cdot 0 \\ (-1)\cdot 1 + 0\cdot 2 & (-1)\cdot 0 + 0\cdot 1 & (-1)\cdot 1 + 0\cdot 0 \\ (-2)\cdot 1 + 1\cdot 2 & (-2)\cdot 0 + 1\cdot 1 & (-2)\cdot 1 + 1\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & 3 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B\cdot A = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}}_{2\times 3} \cdot \underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}}_{3\times 2} =$$ result is a 2×2 matrix

$$= \begin{bmatrix} 1\cdot 3 + 0(-1) + 1(-2) & 1\cdot 2 + 0\cdot 0 + 1\cdot 1 \\ 2\cdot 3 + 1(-1) + 0(-2) & 2\cdot 2 + 1\cdot 0 + 0\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}$$

Example 4 (Order Is Important!)

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Find the matrices AA^T and A^TA.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Check that

$$AA^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \overbrace{\begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}}^{2\times 2}$$

$$A^TA = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}}_{3\times 3}$$

NOTE: Both matrices are SYMMETRIC

Example 5 (Order Is Important!)

Consider the matrices $A = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Find the matrices AB and BA.

$$A = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1(-1) + (-1)(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{1\times 1 \text{ matrix}}$$

$$B \cdot A = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{3\times 3 \text{ matrix}}$$

Powers of a Matrix

If A is a square matrix and k is a positive integer, we define

$$A^k = k\text{th power of } A := A^{k-1}A = A\,A^{k-1} = \underbrace{A\,A\cdots A}_{k \text{ times}}$$

Example 6

Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Find the matrices A^2, A^3, A^4, and A^5.

Have you seen these numbers before? Given $n \geq 1$, can you guess what A^n looks like?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$A^5 = A^4 \cdot A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$$

The numbers that appear in the entries of A, A^2, A^3, A^4, A^5 are

$f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$

$f_5 = 5$, $f_6 = 8, \ldots$ They are the Fibonacci numbers !!!

$$\boxed{f_0 = 0, \quad f_1 = 1 \qquad f_{n+2} = f_{n+1} + f_n \quad \text{fn } n \geq 0}$$

$$A = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} f_3 & f_2 \\ f_2 & f_1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} f_4 & f_3 \\ f_3 & f_2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} f_5 & f_4 \\ f_4 & f_3 \end{bmatrix} \qquad A^5 = \begin{bmatrix} f_6 & f_5 \\ f_5 & f_4 \end{bmatrix} \quad \ldots$$

Thus: $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$

Notice:

$$A^n \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= \begin{bmatrix} f_1 \\ f_0 \end{bmatrix}} = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$$

that is:

$$\boxed{A^n \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}}$$