MA 138 – Calculus 2 with Life Science Applications
Eigenvectors and Eigenvalues
(Section 9.3)

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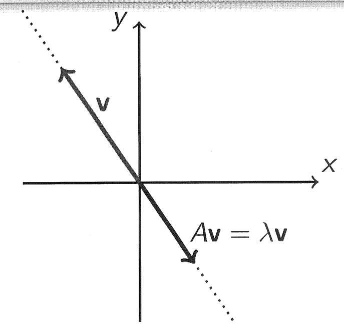
Eigenvalues and Eigenvectors

Definition

Assume that A is a square matrix. A nonzero vector $\mathbf{v}$ that satifies the equation

$$A\mathbf{v} = \lambda\mathbf{v} \qquad (\mathbf{v} \neq \mathbf{0})$$

is an **eigenvector** of the matrix A, and the number λ is an **eigenvalue** of the matrix A.



- The zero vector $\mathbf{0}$ always satisfies the equation $A\mathbf{0} = \lambda\mathbf{0}$ for any choice of λ. Thus $\mathbf{0}$ is not special. That's why we assume $\mathbf{v} \neq \mathbf{0}$.
- The eigenvalue λ can be 0, though.
- **Geometric interpretation**, when the eigenvalue $\lambda \in \mathbb{R}$: If we draw a straight line through the origin in the direction of an eigenvector, then any vector on this straight line will remain on the line after the map A is applied.

$\boxed{\text{Example 1}}$ (Online Homework # 5)

Determine if $\mathbf{v}$ is an eigenvector of the matrix A:

(a) $A = \begin{bmatrix} -35 & -14 \\ 84 & 35 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 19 & 24 \\ -12 & -17 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$;

(c) $A = \begin{bmatrix} -3 & -10 \\ 5 & 12 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$.

(a) $\begin{bmatrix} -35 & -14 \\ 84 & 35 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \end{bmatrix} = -7\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ yes!

(b) $\begin{bmatrix} 19 & 24 \\ -12 & -17 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 86 \\ -58 \end{bmatrix} = 2\begin{bmatrix} 43 \\ -29 \end{bmatrix}$ not an eigenvector.

(c) $\begin{bmatrix} -3 & -10 \\ 5 & 12 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix}$ not an eigenvalue

Example 2 (Online Homework # 6)

Given that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are eigenvectors of the matrix $A = \begin{bmatrix} 28 & 36 \\ -18 & -23 \end{bmatrix}$, determine the corresponding eigenvalues λ_1 and λ_2.

$$\begin{bmatrix} 28 & 36 \\ -18 & -23 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix} = 4\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$\therefore$ eigenvalue is $\underline{\lambda_1 = 4}$

$$\begin{bmatrix} 28 & 36 \\ -18 & -23 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$\therefore$ eigenvalue is $\boxed{\lambda_2 = 1}$

notice $\lambda_1 + \lambda_2 = 1+4 = 28 + (-23) = \text{trace}(A)$

$\lambda_1 \lambda_2 = 1 \cdot 4 = 4 = \det(A)$

$= 28(-23) - (36)(-18)$

Example 3 (Online Homework # 8)

Determine if λ is an eigenvalue of the matrix A:

(a) $A = \begin{bmatrix} 7 & -10 \\ 0 & -3 \end{bmatrix}$ and $\lambda = -2$;

(b) $A = \begin{bmatrix} -4 & -12 \\ 0 & 8 \end{bmatrix}$ and $\lambda = 3$;

(c) $A = \begin{bmatrix} -92 & 42 \\ -196 & 90 \end{bmatrix}$ and $\lambda = -8]$.

(a)

$$\begin{bmatrix} 7 & -10 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \stackrel{?}{=} -2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for some $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We need $\begin{cases} 7v_1 - 10v_2 \stackrel{?}{=} -2v_1 \\ -3v_2 = -2v_2 \end{cases}$

$\Longrightarrow$ $v_2 = 0 \longrightarrow 7v_1 + 2v_1 - 10v_2 = 0$

$\Longrightarrow v_1 = 0$

impossible

$\lambda = -2$ is not an eigenvalue

(b) Is it possible to find $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that

$$\begin{bmatrix} -4 & -12 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \overset{?}{=} 3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} -4v_1 - 12v_2 \overset{?}{=} 3v_1 \\ 8v_2 \overset{?}{=} 3v_2 \end{cases}$$

$$\Longrightarrow v_2 = 0 \Longrightarrow v_1 = 0$$

$\therefore \lambda = 3$ is NOT an eigenvalue

(c) $$\begin{bmatrix} -92 & 42 \\ -196 & 90 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \overset{?}{=} -8\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} -92v_1 + 42v_2 = -8v_1 \\ -196v_1 + 90v_2 = -8v_2 \end{cases}$$

$$\Longrightarrow \begin{cases} -84v_1 + 42v_2 = 0 \\ -196v_1 + 98v_2 = 0 \end{cases}$$

simplify by 42 the first eq. and by 98 the second eq.

$$\begin{cases} -2v_1 + v_2 = 0 \\ -2v_1 + v_2 = 0 \end{cases}$$

hence there is only one equation

$\therefore v_2 = 2v_1$. Any eigenvector corresponding to $\lambda = -8$ is of the form $\begin{bmatrix} t \\ 2t \end{bmatrix}$ or $\boxed{t\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ $t \in \mathbb{R}$

Finding the Eigenvalues

- We are interested in finding $\mathbf{v} \neq \mathbf{0}$ and λ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- We can rewrite this equation as $A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$.
- In order to factor $\mathbf{v}$, we must multiply $\lambda\mathbf{v}$ by the identity matrix I_2.

(In the $n \times n$ case we multiply instead by I_n. The procedure and outcome are exactly the same!)

- Multiplication by I_2 yields $A\mathbf{v} - \lambda I_2\mathbf{v} = \mathbf{0}$.
- We can now factor $\mathbf{v}$, resulting in $(A - \lambda I_2)\mathbf{v} = \mathbf{0}$.
- In Section 9.2, we showed that in order to obtain a nontrivial solution ($\mathbf{v} \neq \mathbf{0}$), the matrix $A - \lambda I_2$ must be singular; that is,

$$\boxed{\det(A - \lambda I_2) = 0}$$

Let us make the previous calculations more explicit (in the 2×2 case):

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}} = \lambda\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}}$$
$$\Updownarrow$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \lambda\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Updownarrow$$
$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Updownarrow$$
$$\underbrace{\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}}_{A-\lambda I_2}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation $\det(A - \lambda I_2) = 0$ that determines the eigenvalues of A is a polynomial equation in λ of degree two. This polynomial is referred to as the **characteristic polynomial** of A.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ define

- $\mathrm{trace}(A) = a + d$,
- $\det(A) = ad - bc$.

The characteristic polynomial has a simple form:

Characteristic Polynomial of the 2×2 Matrix A

$$\det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$
$$\Updownarrow$$
$$(a-\lambda)(d-\lambda) - bc = 0$$
$$\Updownarrow$$
$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0$$

Corollary

If λ_1 and λ_2 are the solutions of the characteristic polynomial, then they must satisfy

$$\mathrm{trace}(A) = \lambda_1 + \lambda_2 \qquad \det(A) = \lambda_1\lambda_2.$$

Example 4 (≈ Problems #49-56, Section 9.3, p. 534)

Consider the matrix $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$.

(a) Find its eigenvalues λ_1 and λ_2 .

(b) Find the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated with the eigenvalues from part (*a*).

(c) Graph the lines through the origin in the direction of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, together with the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and the vectors $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

$$A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$$

(a) $\det(A - \lambda I_2) = 0 \iff$

$$\det \begin{bmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{bmatrix} = (-4-\lambda)(1-\lambda) + 6$$

$$= \lambda^2 + 4\lambda - \lambda - 4 + 6 = \boxed{\lambda^2 + 3\lambda + 2 = 0}$$

$\therefore \quad \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) = 0$

$\therefore \quad \boxed{\lambda_1 = -2}, \quad \boxed{\lambda_2 = -1}$

(b) $\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ find $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Longrightarrow \begin{cases} -4v_1 + 2v_2 = -2v_1 \\ -3v_1 + v_2 = -2v_2 \end{cases} \Longrightarrow \begin{cases} -2v_1 + 2v_2 = 0 \\ -3v_1 + 3v_2 = 0 \end{cases}$

Same equation

$\Longrightarrow \; -v_1 + v_2 = 0$ or $v_1 = v_2$

Hence any eigenvector looks like $\begin{bmatrix} t \\ t \end{bmatrix}$ for $t \in \mathbb{R}$ or $t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Just pick $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\boxed{\lambda_2 = -1}$ we need to solve

$\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = -1\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Longrightarrow \begin{cases} -4w_1 + 2w_2 = -w_1 \\ -3w_1 + w_2 = -w_2 \end{cases}$

$\Longrightarrow \begin{cases} -3w_1 + 2w_2 = 0 \\ -3w_1 + 2w_2 = 0 \end{cases} \Longrightarrow \; w_1 = \frac{2}{3}w_2$

Thus if we pick $w_2 = 3t$ then

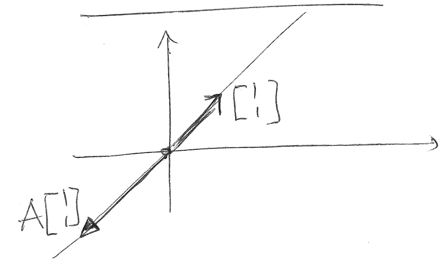
$w_1 = \frac{2}{3}(3t) = 2t$ hence

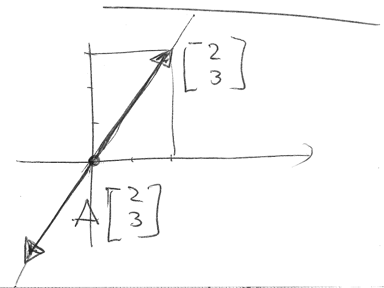
$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t \end{bmatrix} = t\begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ for any } t \in \mathbb{R}$$

Thus choose the simplest eigenvector as $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

THUS:

$\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\begin{bmatrix} 2 \\ 3 \end{bmatrix}$





Example 5 (Online Homework #10)

Find the eigenvalues and associated **unit** eigenvectors of the (symmetric) matrix $A = \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix}$.

$$A = \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix}$$

$$\det(A - \lambda I_2) = \det \begin{bmatrix} 5-\lambda & -10 \\ -10 & 20-\lambda \end{bmatrix} =$$

$$= (5-\lambda)(20-\lambda) - 100 = 0$$

$$\iff \lambda^2 - 25\lambda + \cancel{100} - \cancel{100} = 0$$

$$\iff \boxed{\lambda^2 - 25\lambda = 0} \iff \boxed{\lambda_1 = 0 \quad \lambda_2 = 25}$$

(1) $\boxed{\lambda_1 = 0}$ $\quad \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ hence

$$\begin{cases} 5v_1 - 10v_2 = 0 \\ -10v_1 + 20v_2 = 0 \end{cases} \iff \begin{cases} v_1 - 2v_2 = 0 \\ v_1 - 2v_2 = 0 \end{cases}$$

Hence $\boxed{v_1 = 2v_2}$ Set $v_2 = t \in \mathbb{R}$ then $v_1 = 2t$. Hence any eigenvector for $\lambda = 0$ looks like $\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

Pick e.g. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ it has length $\sqrt{4+1} = \sqrt{5}$

If we take $\boxed{\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ this has length 1 (unit eigenvector)

(2) $\boxed{\lambda_2 = 25}$ We need $\begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 25 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$\iff \begin{cases} 5w_1 - 10w_2 = 25w_1 \\ -10w_1 + 20w_2 = 25w_2 \end{cases} \iff \begin{cases} -20w_1 - 10w_2 = 0 \\ -10w_1 - 5w_2 = 0 \end{cases}$$

$$\iff \{ 2w_1 + w_2 = 0 \quad \text{same equation}$$

i.e. $w_2 = -2w_1$ Pick for example $w_1 = t \in \mathbb{R}$ then $w_2 = -2t$ so $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $t=1$ we get $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ but it has length $\sqrt{1^2 + (-2)^2} = \sqrt{5}$. Thus divide it by its length and we get an eigenvector of length 1

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example 6 (Online Homework #12)

Let $A = \begin{bmatrix} -4 & 3 \\ 5 & k \end{bmatrix}$.

Find the value of k so that A has 0 as an eigenvalue.

We know that for a 2×2 matrix A with eigenvalues λ_1 and λ_2 then

$\det(A) = \lambda_1\lambda_2$ and $\operatorname{trace}(A) = \lambda_1 + \lambda_2$

Since we want one of the 2 eigenvalues to be 0 then $\det(A) = 0$.

$$\det\begin{bmatrix} -4 & 3 \\ 5 & k \end{bmatrix} = 0 \iff -4k - 3\cdot 5 = 0$$

$$\iff \boxed{k = -15/4}$$

Example 7 (Online Homework #13)

For which value of k does the matrix $A = \begin{bmatrix} -3 & k \\ -8 & -8 \end{bmatrix}$ have one real eigenvalue of multiplicity 2?

$A = \begin{bmatrix} -3 & k \\ -8 & -8 \end{bmatrix}$. To find the eigenvalues of A

$$\det(A - \lambda I_2) = 0 \implies$$

$$\det\begin{bmatrix} -3-\lambda & k \\ -8 & -8-\lambda \end{bmatrix} = (-3-\lambda)(-8-\lambda) + 8\cdot k = 0$$

$$\implies \lambda^2 + 11\lambda + 24 + 8k = 0$$

$$\lambda_{1,2} = \frac{-11 \pm \sqrt{11^2 - 4(1)(24+8k)}}{2\cdot 1}$$

To have $\lambda_1 = \lambda_2$ we need the discriminant of the equation to be 0:

$$121 - 96 - 32k = 0 \qquad \boxed{k = \frac{25}{32}}$$

Example 8 (Online Homework #16)

Find a matrix A such that $\mathbf{v}_1 = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ are eigenvectors of A, with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ respectively.

Want $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

(1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -3 \\ -4 \end{bmatrix} = 5\begin{bmatrix} -3 \\ -4 \end{bmatrix}$ $\boxed{\text{AND}}$

(2) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = -1\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

So we must have from (1) and (2)

$\begin{cases} -3a - 4b = -15 \\ -3c - 4d = -20 \end{cases}$ and $\begin{cases} 2a - 5b = -2 \\ 2c - 5d = 5 \end{cases}$

OR

$\begin{cases} -3a - 4b = -15 \\ 2a - 5b = -2 \end{cases}$ $\begin{cases} -3c - 4d = -20 \\ 2c - 5d = 5 \end{cases}$

$\begin{bmatrix} -3 & -4 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -15 \\ -2 \end{bmatrix}$ $\begin{bmatrix} -3 & -4 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -20 \\ 5 \end{bmatrix}$

same matrix

its inverse is

$\frac{1}{23}\begin{bmatrix} -5 & 4 \\ -2 & -3 \end{bmatrix}$ so that

$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{23}\begin{bmatrix} -5 & 4 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -15 \\ -2 \end{bmatrix} = \begin{bmatrix} 67/23 \\ 36/23 \end{bmatrix}$

$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{23}\begin{bmatrix} -5 & 4 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -20 \\ 5 \end{bmatrix} = \begin{bmatrix} 120/23 \\ 25/23 \end{bmatrix}$

$\therefore \boxed{A = \begin{bmatrix} 67/23 & 36/23 \\ 120/23 & 25/23 \end{bmatrix}}$

Example 9 (Complex Eigenvalues)

Consider the matrix $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$.

(a) Find its eigenvalues.

(b) Find the eigenvectors associated with the eigenvalues from part (*a*).

$$A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$ has eigenvalues given by

$$\det\left[A - \lambda I_2\right] = 0 \iff \det\begin{bmatrix} \sqrt{3}-\lambda & -1 \\ 1 & \sqrt{3}-\lambda \end{bmatrix} = 0$$

$$\iff (\sqrt{3}-\lambda)^2 + 1 = 0 \iff$$

$$3 - 2\sqrt{3}\lambda + \lambda^2 + 1 = 0 \iff \boxed{\lambda^2 - 2\sqrt{3}\lambda + 4 = 0}$$

Quadratic equation: $\lambda_{1,2} = \dfrac{2\sqrt{3} \pm \sqrt{12-16}}{2}$

$$\lambda_{1,2} = \frac{2\sqrt{3} \pm 2\sqrt{-1}}{2} = \boxed{\sqrt{3} \pm i}$$

$\boxed{\lambda_1 = \sqrt{3} + i}$ To find the eigenvector we need

$$\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (\sqrt{3}+i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\iff \begin{cases} \sqrt{3}v_1 - v_2 = (\sqrt{3}+i)v_1 \\ v_1 + \sqrt{3}v_2 = (\sqrt{3}+i)v_2 \end{cases}$$

$$\iff \begin{cases} \cancel{\sqrt{3}v_1} - v_2 = \cancel{\sqrt{3}v_1} + iv_1 \\ v_1 + \cancel{\sqrt{3}v_2} = \cancel{\sqrt{3}v_2} + iv_2 \end{cases}$$

$$\iff \begin{cases} v_2 = -iv_1 \\ v_1 = iv_2 \end{cases}$$

Notice that they are the same equation!!!

$v_1 = iv_2 \Rightarrow iv_1 = i^2 v_2 = -v_2$

Hence pick for example the second equation $\boxed{v_1 = iv_2}$ If $v_2 = t \in \mathbb{R}$

then $v_1 = it$ so $\begin{bmatrix} it \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$

the simplest eigenvector is $\begin{bmatrix} i \\ 1 \end{bmatrix}$

For $\boxed{\lambda_2 = \sqrt{3} - i}$ we need

$$\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (\sqrt{3}-i)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \iff$$

$$\begin{cases} \sqrt{3}w_1 - w_2 = (\sqrt{3}-i)w_1 \\ w_1 + \sqrt{3}w_2 = (\sqrt{3}-i)w_2 \end{cases} \iff \begin{cases} w_2 = iw_1 \\ w_1 = -iw_2 \end{cases}$$

Pick eg.: $w_1 = -iw_2$ so $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$

So notice that

$\lambda_1 = \sqrt{3} + i$ $\qquad$ $\lambda_2 = \sqrt{3} - i$

$\underline{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ $\qquad$ $\underline{w} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

they are complex conjugate of each other!