MA 138 – Calculus 2 with Life Science Applications
Limits and Continuity
(Section 10.2)

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Informal Definition of Limits

We need to extend the notion of limits and continuity to the multivariable setting. The ideas are the same as in the one-dimensional case. We will discuss only the two-dimensional case, but note that everything in this section can be generalized to higher dimensions. Let's start with an informal definition of limits.

Informal Definition of Limits

We say that the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) is equal to L if $f(x, y)$ can be made arbitrarily close to L whenever the point (x, y) is sufficiently close (but not equal) to the point (x_0, y_0). We denote this concept by

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

Limit Laws for Functions of Two Variables

Suppose c is a constant and the limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y)$$

exist. Then the following properties hold:

- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \Big[f(x, y)+g(x, y)\Big] = \Big[\lim_{(x,y)\to(x_0,y_0)} f(x, y)\Big] + \Big[\lim_{(x,y)\to(x_0,y_0)} g(x, y)\Big]$
- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \Big[c\, f(x, y)\Big] = c\Big[\lim_{(x,y)\to(x_0,y_0)} f(x, y)\Big]$
- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \Big[f(x, y)\cdot g(x, y)\Big] = \Big[\lim_{(x,y)\to(x_0,y_0)} f(x, y)\Big]\cdot\Big[\lim_{(x,y)\to(x_0,y_0)} g(x, y)\Big]$
- $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)}{\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x, y)}$

  provided $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x, y) \neq 0.$

Example 1 (≈ Problems #2, 4, 10, Section 10.2, p. 581)

Use the properties of limits to calculate the following limits

- $\displaystyle\lim_{(x,y)\to(-1,1)} 2xy + 3x^2$
- $\displaystyle\lim_{(x,y)\to(1,-2)} (2x^3 - 3y)(xy - 2)$
- $\displaystyle\lim_{(x,y)\to(1,-2)} \frac{2x^2 + y}{2xy + 3}$

(*) $\lim_{(x,y)\to(-1,1)} (2xy + 3x^2) = \left[\lim_{(x,y)\to(-1,1)} (2xy)\right] + \left[\lim_{(x,y)\to(-1,1)} (3x^2)\right]$

$= 2\left[\lim_{(x,y)\to(-1,1)} (x)\right]\left[\lim_{(x,y)\to(-1,1)} (y)\right] + 3\left[\lim_{(x,y)\to(-1,1)} x\right]^2$

$= 2(-1)(1) + 3(-1)^2 = -2+3 = \boxed{1}$

(*) $\lim_{(x,y)\to(1,-2)} (2x^3 - 3y)(xy-2) =$

$= \left[\lim_{(x,y)\to(1,-2)} (2x^3-3y)\right]\left[\lim_{(x,y)\to(1,-2)} (xy-2)\right]$

$= \cdots = (2(1)^3 - 3(-2))(1(-2)-2) = (2+6)(-2-2)$

$= 8(-4) = \boxed{-32}$

(*) $$\lim_{(x,y)\to(1,-2)} \frac{2x^2+y}{2xy+3} = \frac{\lim_{(x,y)\to(1,-2)} (2x^2+y)}{\lim_{(x,y)\to(1,-2)} (2xy+3)}$$

$$= \frac{2\left[\lim_{(x,y)\to(1,-2)} x\right]^2 + \left[\lim_{(x,y)\to(1,-2)} y\right]}{2\left[\lim_{(x,y)\to(1,-2)} x\right]\left[\lim_{(x,y)\to(1,-2)} y\right] + 3}$$

$$= \frac{2(1)^2 + (-2)}{2(1)(-2)+3} = \frac{2-2}{-4+3} = \frac{0}{-1} = \boxed{0}$$

Limits That Do Not Exist

In the one-dimensional case, there were only two ways in which we could approach a number: from the left or from the right. If the two limits were different, we said that the limit did not exist. In two dimensions, there are many more ways that we can approach the point (x_0, y_0), namely, by any curve in the xy-plane that ends up at the point (x_0, y_0). We call such curves paths.

Suppose that

- $f(x,y)$ approaches L_1 as (x,y) approaches (x_0,y_0) along path C_1,
- $f(x,y)$ approaches L_2 as (x,y) approaches (x_0,y_0) along path C_2,
- $L_1 \neq L_2$,

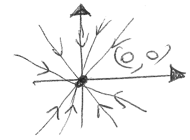
then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Example 2 (Example 3, Section 10.2, p. 577)

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ along paths of the form $y = mx$.

What does this say about the limit?

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{along curves } y=mx$$



$$= \lim_{\substack{x\to 0\\ y=mx}} \frac{x^2-(mx)^2}{x^2+(mx)^2} = \lim_{\substack{x\to 0\\ y=mx}} \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)}$$

$$= \lim_{\substack{x\to 0\\ y=mx}} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}$$

This means that the limit depends on the slope (m) of the line we choose to approach (0,0).

Hence the limit DOES NOT exist

Remark about Example 2

The level curves of the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ are of the form

$$\frac{x^2-y^2}{x^2+y^2} = c \iff x^2-y^2 = c(x^2+y^2)$$

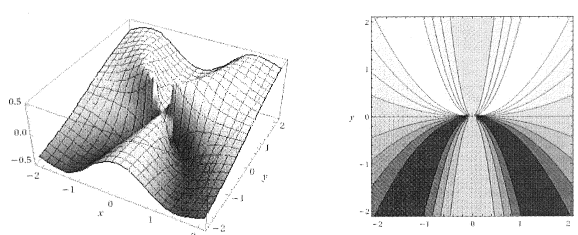
$$\iff x^2(1-c) = (1+c)y^2 \iff y^2 = \frac{1-c}{1+c}x^2$$

$$\iff y = \pm\underbrace{\sqrt{\frac{1-c}{1+c}}}_{m}\,x$$

That is, the level curves are straight lines through the origin (i.e., $y = mx$).

Example 3 (Problem # 3, Exam 3, Spring '13)

The graph and level curves of the function $f(x,y) = \dfrac{x^2y}{x^4+y^2}$ are shown below



- Evaluate the limit $\displaystyle\lim_{(x,y)\to(1,-3)} \frac{x^2y}{x^4+y^2}$.
- Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ exist? Explain.

(**Hint:** as the picture above suggests, the level curves of $f(x,y)$ are parabolas in the xy-plane of the form $y = mx^2$, where m is any constant.)

$$f(x,y) = \frac{x^2y}{x^4+y^2}$$

(*) $$\lim_{(x,y)\to(1,-3)} \frac{x^2y}{x^4+y^2} = \frac{\left[\lim_{(x,y)\to(1,-3)} x\right]^2\left[\lim_{(x,y)\to(1,-3)} y\right]}{\left[\lim_{(x,y)\to(1,-3)} x\right]^4 + \left[\lim_{(x,y)\to(1,-3)} y\right]^2}$$

$$= \frac{(1)^2(-3)}{(1)^4+(-3)^2} = \frac{-3}{10} = \boxed{-0.3}$$

(**) $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \frac{0}{0} \quad \text{so what}$$

can we do? Consider limits along parabolas

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx^2}} \frac{x^2y}{x^4+y^2} = \lim_{\substack{x\to 0\\ y=mx^2}} \frac{x^2(mx^2)}{x^4+(mx^2)^2}$$

$$= \lim_{\substack{x\to 0\\ y=mx^2}} \frac{x^4\cdot m}{x^4+m^2x^4} = \lim_{\substack{x\to 0\\ y=mx^2}} \frac{\cancel{x^4}\cdot m}{\cancel{x^4}(1+m^2)}$$

$$= \frac{m}{1+m^2}$$

This says that the limit depends on the amplitude (m) of the parabola that we choose to approach $(0,0)$. Hence the limit DOES NOT exist.

Remark about Example 3

The level curves of the function $f(x,y) = \dfrac{x^2y}{x^4+y^2}$ are of the form

$$\frac{x^2y}{x^4+y^2} = c \iff x^2y = c(x^4+y^2)$$

$$\iff (x^2)^2 - \frac{y}{c}x^2 + y^2 = 0 \iff x^2 = \frac{y/c \pm \sqrt{(y/c)^2 - 4y^2}}{2}$$

$$\iff x^2 = \frac{y \pm y\sqrt{1-4c^2}}{2c} \iff y = \underbrace{\frac{2c}{1\pm\sqrt{1-4c^2}}}_{m} x^2$$

That is, the level curves are parabolas through the origin (i.e., $y = mx^2$).

Example 4 (≈ Example 4, Section 10.2, p. 514)

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{xy+y^3}$

- along paths of the form $y = mx$;
- along paths of the form $x = my^2$;

Does the limit exist?

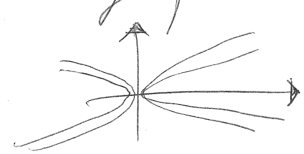
(*) $$\lim_{(x,y)\to(0,0)} \frac{4xy}{xy+y^3} = \frac{4(0)(0)}{0\cdot 0+0^3} = \frac{0}{0}$$

If we approach $(0,0)$ along $y=mx$ we get

$$\lim_{\substack{x\to 0\\ y=mx}} \frac{4x(mx)}{x(mx)+(mx)^3} = \lim_{\substack{x\to 0\\ y=mx}} \frac{4mx^2}{x^2m+m^3x^3}$$

$$= \lim_{\substack{x\to 0\\ y=mx}} \frac{4m}{m+mx^3} = \frac{4m}{m} = 4$$

(**) However, let us approach along parabolas of the form $x = my^2$



$$\lim_{\substack{(x,y)\to 0\\ x=my^2}} \frac{4xy}{xy+y^3} = \lim_{\substack{y\to 0\\ x=my^2}} \frac{4(my^2)y}{(my^2)y+y^3}$$

$$= \lim_{\substack{y\to 0\\ x=my^2}} \frac{\cancel{y^3}(4m)}{\cancel{y^3}(m+1)} = \frac{4m}{m+1}$$

We see that now the limit depends on the amplitude m of the parabola along which we approach the origin $(0,0)$.

Hence the limit <u>DOES NOT</u> exist.

Remark about Example 4

The level curves of the function $f(x,y) = \dfrac{4xy}{xy+y^3}$ are of the form

$$\frac{4xy}{xy+y^3} = c \iff 4xy = c(xy+y^3)$$

$$\iff cy^3 - 4xy + cxy = 0 \iff y[cy^2 - (4-c)x] = 0$$

$$\iff y = 0 \quad \text{or} \quad x = \underbrace{\frac{c}{4-c}}_{m} y^2$$

That is, the level curves are parabolas through the origin and symmetric with respect to the x-axis (i.e., $x = my^2$).

Continuity

This notion is analogous to that in the one-dimensional case.

Continuity

A function $f(x,y)$ is **continuous at a point** (x_0,y_0) if

1. $f(x,y)$ is defined at (x_0,y_0);
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists;
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

We say f is **continuous** on D if f is continuous at every point of D.

Using this definition and the Limit Laws, one can show that functions defined by polynomials in two variables (i.e., expressions that are sums of terms of the form $\alpha x^n y^m$) are continuous on $\mathbb{R}^2$. Rational functions (i.e., ratio of polynomials) are also continuous everywhere on their domain.

Example 5 (Problem #23, Section 10.2, p. 581)

Show that

$$f(x,y) = \begin{cases} \dfrac{4xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is discontinuous at $(0,0)$.

the function $f(x,y) = \frac{4xy}{x^2+y^2}$ is continuous for all $(x,y) \neq (0,0)$ as it is the quotient of 2 polynomials that are defined for x and y not equal to 0.

However, if we compute the limit as (x,y) approach $(0,0)$ along lines $y=mx$, we obtain

$$\lim_{\substack{x\to 0\\ y=mx}} \frac{4xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=mx}} \frac{4x\,(mx)}{x^2+(mx)^2} = \lim_{x\to 0} \frac{4m}{1+m^2}$$

$$= \frac{4m}{1+m^2}$$

Since the limiting value depends on m, the limit DOES NOT EXIST

Hence $f(x,y)$ is not continuous at $(0,0)$ because the value $f(0,0)$ is not equal to the limit, as the limit does not exist.

Example 6

Show that $g(x,y) = \dfrac{4x^2y}{x^2+y^2}$ is continuous for every (x,y) in $\mathbb{R}^2$.

As in Example 5, the function $g(x,y) = \frac{4x^2y}{x^2+y^2}$ is continuous for all $(x,y) \neq (0,0)$ as it is the quotient of 2 polynomial functions, which are continuous for $(x,y) \neq (0,0)$. The issue is at $(0,0)$

we need to argue that $\lim_{(x,y)\to(0,0)} g(x,y)$ exists and is equal to 0.

Observe that $x^2 \leq x^2+y^2$ for all (x,y)

Hence $\quad 0 \leq \dfrac{x^2}{x^2+y^2} \leq 1$

$$\Longrightarrow \quad 0 \leq \left| \frac{4y\,x^2}{x^2+y^2} \right| \leq |4y|$$

By the Sandwich theorem when $(x,y) \to (0,0)$ the functions on the outside converge to 0

Hence the function in the middle converge to 0 so $\boxed{\lim_{(x,y)\to(0,0)} \dfrac{4x^2y}{x^2+y^2} = 0}$

Composition of Continuous Functions is Continuous

Suppose $f : D \longrightarrow \mathbb{R}$, with $D \subset \mathbb{R}^2$, and $g : I \longrightarrow \mathbb{R}$, with I a subset of $\mathbb{R}$ containing the range of f. Then the composition $(g \circ f)(x, y)$ is defined as the function $h : D \longrightarrow \mathbb{R}$

$$h(x, y) = (g \circ f)(x, y) = g[f(x, y)].$$

If f is continuous at (x_0, y_0) and g is continuous at $z = f(x_0, y_0)$, then one can show that $h(x, y) = (g \circ f)(x, y) = g[f(x, y)]$ is continuous at (x_0, y_0).

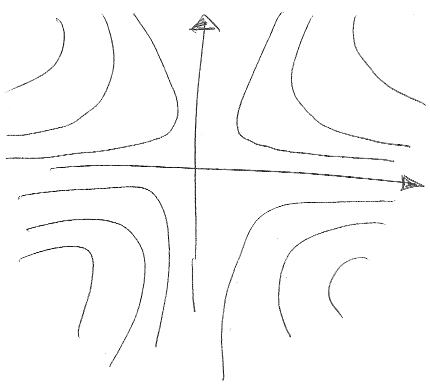
Example

Consider the function $h(x, y) = e^{x^2+y^2}$. If we set $z = f(x, y) = x^2 + y^2$ and $g(z) = e^z$, then we obtain

$$h(x, y) = g[f(x, y)].$$

Since $f(x, y)$ is continuous for all $(x, y) \in \mathbb{R}^2$ and $g(z)$ is continuous for all z in the range of $f(x, y)$, then h is continuous for all $(x, y) \in \mathbb{R}^2$.

I should show the graph of $g(x,y)$ and its level curves



$$\frac{4x^2y}{x^2+y^2} = c$$

$$\Longleftrightarrow$$

$$4x^2y = cx^2 + cy^2$$

$$x^2(4y-c) = cy^2$$

$$x^2 = \frac{cy^2}{4y-c} \Longleftrightarrow x = \pm\frac{\sqrt{c}\,y}{\sqrt{4y-c}}$$