MA 138 – Calculus 2 with Life Science Applications
Tangent Planes, Differentiability, and Linearization
(Section 10.4)

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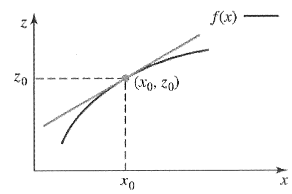
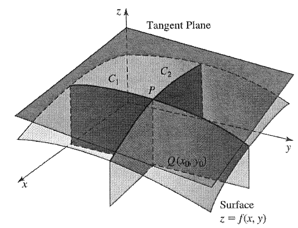
- The key idea in both the one- and the two-dimensional case is to approximate functions by linear functions, so that the error in the approximation vanishes as we approach the point at which we approximated the function.
- If $z = f(x)$ is differentiable at $x = x_0$, then the equation of the tangent line of $z = f(x)$ at (x_0, z_0) with $z_0 = f(x_0)$ is given by

$$z - z_0 = f'(x_0)(x - x_0).$$



- We now generalize this situation to functions of two variables. The analogue of a tangent line is called a **tangent plane**, an example of which is shown in the picture on the right.



Tangent Plane

- Let $z = f(x, y)$ be a function of two variables.
- We saw that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, evaluated at (x_0, y_0), are the slopes of tangent lines at the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$, to certain curves through (x_0, y_0, z_0) on the surface $z = f(x, y)$.
- These two tangent lines, one in the x-direction, the other in the y-direction, define a unique plane.
- If, in addition, $f(x, y)$ has partial derivatives that are continuous on an open disk containing (x_0, y_0), then we can show that the tangent line at (x_0, y_0, z_0) to any other smooth curve on the surface $z = f(x, y)$ through (x_0, y_0, z_0) is contained in this plane.
- The plane is then called the tangent plane.

More precisely, one can show the following result:

Equation of the Tangent Plane

If the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, **exists**, then that tangent plane has the equation

$$z - z_0 = \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0).$$

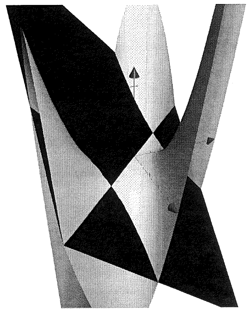
- We should observe the similarity of this equation to the equation of the tangent line in the one-dimensional case.
- As we mentioned, the mere existence of the partial derivatives $\dfrac{\partial f(x_0, y_0)}{\partial x}$ and $\dfrac{\partial f(x_0, y_0)}{\partial y}$ is not enough to guarantee the existence of a tangent plane at (x_0, y_0, z_0); something stronger is needed.

Example 1

Find an equation of the tangent plane to surface given by the graph of the function

$$z = f(x,y) = xy^2 + x^2y$$

at the point $(1,-1,0)$.



$\boxed{f(x,y) = xy^2 + x^2y}$ $\qquad$ $P(1,-1,0)$

notice that $f(1,-1) = 1(-1)^2 + (1)^2(-1) = 1-1=0$

$f_x = y^2 + 2xy$ $\quad$ so $f_x(1,-1) = (-1)^2 + 2(1)(-1) = \boxed{-1}$

$f_y = 2xy + x^2$ $\quad$ so $f_y(1,-1) = 2(1)(-1) + 1^2 = \boxed{-1}$

Thus the equation of the tangent plane is

$$z - 0 = (-1)(x-1) + (-1)(y-(-1))$$

or $\boxed{z = -x - y}$

Example 2 (Problem #4, Online Homework)

Find an equation of the tangent plane to surface given by the graph of the function

$$F(r,s) = r^4 s^{-0.5} - s^{-4}$$

at the point with $r_0 = 1$ and $s_0 = 1$.

$F(r,s) = r^4 s^{-0.5} - s^{-4}$

$F(1,1) = 1^4 \cdot (1)^{-0.5} - (1)^{-4} = 1-1 = \boxed{0}$

$F_r = \frac{\partial F}{\partial r} = 4r^3 s^{-0.5}$ $\qquad$ $\boxed{F_r(1,1) = 4}$

$F_s = \frac{\partial F}{\partial s} = r^4(-0.5 s^{-1.5}) - (-4)s^{-5}$

$= -0.5 r^4 s^{-1.5} + 4s^{-5}$

$\boxed{F_s(1,1) = -0.5 + 4 = 3.5}$ Thus the equation of the tangent plane is

$$z - 0 = 4(r-1) + 3.5(s-1)$$

or $\boxed{z = 4r + 3.5s - 7.5}$ ///

Review of differentiability for a function of one variable

If $z = f(x)$ is a function of one variable, the tangent line is used to approximate $f(x)$ at $x = x_0$. The linearization of $f(x)$ at $x = x_0$ is given by

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

The distance between $f(x)$ and its linear approximation at $x = x_0$ is then

$$|f(x) - L(x)| = |f(x) - f(x_0) - f'(x_0)(x - x_0)|.$$

If we divide the latter equation by the distance $|x - x_0|$, we find that

$$\left|\frac{f(x) - L(x)}{x - x_0}\right| = \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right|.$$

Taking a limit and using the definition of the derivative at $x = x_0$, yields

$$\lim_{x \to x_0} \left|\frac{f(x) - L(x)}{x - x_0}\right| = 0.$$

We say that $f(x)$ is differentiable at $x = x_0$ if the above equation holds.

Differentiability and Linearization

Suppose that $f(x, y)$ is a function of two independent variables with both $\partial f/\partial x$ and $\partial f/\partial y$ defined on an open disk containing (x_0, y_0).

- Set $\quad L(x, y) = f(x_0, y_0) + \dfrac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \dfrac{\partial f(x_0, y_0)}{\partial y}(y - y_0).$
- $f(x, y)$ is differentiable at (x_0, y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \left|\frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}\right| = 0.$
- If $f(x, y)$ is differentiable at (x_0, y_0), then $z = L(x, y)$ provides the equation of the tangent plane to the graph of f at (x_0, y_0, z_0).
- $f(x, y)$ is differentiable if it is differentiable at every point of its domain.
- Suppose f is differentiable at (x_0, y_0), the approximation $f(x, y) \approx L(x, y)$ is the **standard linear approximation**, or the tangent plane approximation, of $f(x, y)$ at (x_0, y_0).

- That $f(x, y)$ is differentiable at (x_0, y_0) means that the function $f(x, y)$ is close to the tangent plane at (x_0, y_0) for all (x, y) close to (x_0, y_0).
- As in the one-dimensional case, the following theorem holds:

Theorem

If $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

- The mere existence of the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ at (x_0, y_0), however, is not enough to guarantee differentiability (and, consequently, the existence of a tangent plane at a certain point).
- The following differentiability criterion suffices for all practical purposes.

Sufficient Condition For Differentiability

Suppose $f(x, y)$ is defined on an open disk centered at (x_0, y_0) and the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are continuous on an open disk centered at (x_0, y_0). Then $f(x, y)$ is differentiable at (x_0, y_0).

Example 3 (Problem #6, Online Homework)

Estimate $f(3.01, 2.02)$ given that

$$f(3, 2) = 4 \qquad f_x(3, 2) = -5 \qquad f_y(3, 2) = 2.$$

The linearization of $f(x,y)$ at $(3,2)$ is:

$$\boxed{L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)}$$

given that $f(3,2)=4$, $f_x(3,2)=-5$, and $f_y(3,2)=2$ we get:

$$\boxed{L(x,y) = 4 - 5(x-3) + 2(y-2)}$$

Hence:

$$f(3.01, 2.02) \cong L(3.01, 2.02) = 4 - 5(0.01) + 2(0.02)$$

$$= 4 - 0.05 + 0.04 = 4 - 0.01 = \boxed{3.99}$$

Example 4 (Problem #5(b), Exam 4, Spring 2012)

Find the linear approximation of the function

$$f(x,y) = x \cdot e^{xy}$$

at $(1,0)$, and use it to approximate $f(1.1,-0.1)$. Using a calculator, compare the approximation with the exact value of $f(1.1,-0.1)$.

$$f(x,y) = x e^{xy} \qquad P(\underset{1}{x_0}, \underset{0}{y_0}, \underset{1}{z_0})$$

$$z_0 = f(1,0) = 1 \cdot e^{1\cdot 0} = \boxed{1}$$

$$f_x = 1 \cdot e^{xy} + x \cdot e^{xy} \cdot y = e^{xy}(1+xy)$$

$$f_x(1,0) = e^0(1+1\cdot 0) = \boxed{1}$$

$$f_y = x(e^{xy} \cdot x) = x^2 e^{xy} \qquad f_y(1,0) = 1 \cdot e^0 = \boxed{1}$$

Hence the linear approx at $(1,0)$ is

$$\boxed{L(x,y) = 1 + 1(x-1) + 1\cdot(y-0) = 1 + x - 1 + y = x+y}$$

$$f(1.1,-0.1) \cong L(1.1,-0.1) = \boxed{1}$$

actual value $\boxed{1.1 e^{-0.11} = 0.9854}$

Example 5 (Problem #9, Online Homework)

Find the linearization of the function

$$f(x,y) = \sqrt{23 - x^2 - 5y^2}$$

at the point $(-3,-1)$.
Use the linear approximation to estimate the value of $f(-3.1,-0.9)$.

$$f(x,y) = \sqrt{23 - x^2 - 5y^2} = (23 - x^2 - 5y^2)^{1/2}$$

$$f(-3,-1) = \sqrt{23 - 9 - 5(1)} = \sqrt{9} = \underline{\underline{3}}$$

$$f_x = \frac{1}{2}(23 - x^2 - 5y^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{23 - x^2 - 5y^2}}$$

$$f_y = \frac{1}{2}(23 - x^2 - 5y^2)^{-1/2}(-10y) = \frac{-5y}{\sqrt{23 - x^2 - 5y^2}}$$

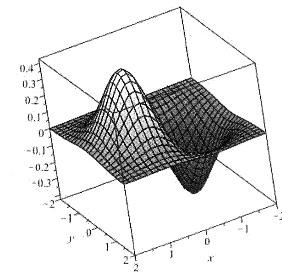
$$f_x(-3,-1) = \frac{-(-3)}{3} = \boxed{1} \qquad f_y(-3,-1) = \frac{-5(-1)}{3} = \boxed{5/3}$$

$$\therefore\ \underline{L(x,y)} = 3 + 1(x+3) + \tfrac{5}{3}(y+1) = \underline{x + \frac{5}{3}y + \frac{23}{3}}$$

$$\therefore\ f(-3.1,-0.9) \cong L(-3.1,-0.9) = 3 - 0.1 + \frac{5}{3}0.1 = 3 + \frac{2}{30} \cong \underline{3.0666}$$

Example 6 (Problem #6, Exam 3, Spring 2013)

Consider the function $f(x,y) = x\,e^{-x^2-y^2}$ whose graph is given in the picture on the right.



(a) Find the z-coordinate z_0 of the point P on the graph of the function $f(x,y)$ with x-coordinate $x_0 = 1$ and y-coordinate $y_0 = 1$.

(b) Write the equation of the tangent plane to the graph of the function $f(x,y)$ at the point P, as above, with coordinates $x_0 = 1$ and $y_0 = 1$.

(c) Write the linear approximation, $L(x,y)$, of the function f at the point with $x_0 = 1$ and $y_0 = 1$, as above, and use it to approximate $f(1.1, 0.9)$.
Compare this approximate value to the exact value $f(1.1, 09)$.

$$f(x,y) = x \cdot e^{-x^2-y^2}$$

(a) $f(1,1) = z_0 = 1 \cdot e^{-2} = \boxed{e^{-2}}$

(b) $f_x = e^{-x^2-y^2} + x\,e^{-x^2-y^2}(-2x) = \underline{e^{-x^2-y^2}(1 - 2x^2)}$

$\boxed{f_x(1,1) = -e^{-2}}$

$f_y = x\,e^{-x^2-y^2}(-2y) = \underline{-2xy\,e^{-x^2-y^2}}$

$\boxed{f_y(1,1) = -2e^{-2}}$

eq. of tg plane: check that:

$$z = e^{-2}[4 - x - 2y]$$

(c) $L(x,y) = e^{-2}(4 - x - 2y)$ $\quad f(1.1, 0.9) \approx L(1.1, 0.9) = 0.1488$

exact value of $f(1.1, 0.9) = \underline{0.1459}$

Functions of more than two variables

Similar discussions can be carried for functions of more than two variables.

For example, if $w = f(x,y,z)$ is a function of three independent variables which is differentiable at a point (x_0, y_0, z_0), then the linear approximation $L(x,y,z)$ of f at (x_0, y_0, z_0) is given by the formula

$$L(x,y,z) =$$
$$f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)\cdot(x - x_0) + f_y(x_0,y_0,z_0)\cdot(y - y_0) + f_z(x_0,y_0,z_0)\cdot(z - z_0).$$

Example 7 (Problem #10, Online Homework)

Find the linear approximation to the function

$$f(x, y, z) = \frac{xy}{z}$$

at the point $(-2, -3, -1)$.

$f(x,y,z) = \frac{xy}{z}$ $\qquad f(-2,-3,-1) = \frac{(-2)(-3)}{(-1)} = \boxed{-6}$

$f_x(x,y,z) = \frac{y}{z}$ $\qquad f_x(-2,-3,-1) = \frac{-3}{-1} = \boxed{3}$

$f_y(x,y,z) = \frac{x}{z}$ $\qquad f_y(-2,-3,-1) = \frac{-2}{-1} = \boxed{2}$

$f_z(x,y,z) = -\frac{xy}{z^2}$ $\qquad f_z(-2,-3,-1) = -\frac{(-2)(-3)}{(-1)^2} = \boxed{-6}$

the linear approximation is

$$\boxed{L(x,y,z) = -6 + 3(x+2) + 2(y+3) - 6(z+1)}$$

or $\underline{L(x,y,z) = 3x + 2y - 6z}$ after simplifying