

# MA 138 – Calculus 2 with Life Science Applications

## Vector Valued Functions

(Section 10.4)

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We will encounter vector-valued functions where  $n = m = 2$  in Chapter 11.

### Example

As an example, consider a community consisting of two species.

Let  $u$  and  $v$  denote the respective densities of the species and  $f(u, v)$  and  $g(u, v)$  the per capita growth rates of the species as functions of the densities  $u$  and  $v$ .

We can then write this relationship as a map

$$\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (u, v) \mapsto \begin{bmatrix} f(u, v) \\ g(u, v) \end{bmatrix}.$$

**E.g.**, in the Lotka-Volterra predator-prey model:  $(u, v) \mapsto \begin{bmatrix} \alpha - \beta v \\ \gamma u - \delta \end{bmatrix}$ ,

where  $\alpha, \beta, \gamma$ , and  $\delta$  are constants.

## Vector-valued functions

- So far, we have considered only real-valued functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .
- We now extend our discussion to functions whose the range is a subset of  $\mathbb{R}^m$  — that is,  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .
- Such functions are vector-valued functions, since they take on values that are represented by vectors:

$$\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad (x_1, x_2, \dots, x_n) \mapsto \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_m(x_1, x_2, \dots, x_n) \end{bmatrix}.$$

- Here, each function  $f_i(x_1, \dots, x_n)$  is a real-valued function:

$$f_i : \mathbb{R}^n \rightarrow \mathbb{R} \quad (x_1, x_2, \dots, x_n) \mapsto f_i(x_1, x_2, \dots, x_n).$$

## Review

- We have defined earlier the linearization at a point  $(x_0, y_0)$  of a real-valued function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ; namely,

$$L_f(x, y) = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0).$$

- We can write the above equation in matrix notation as

$$L_f(x, y) = f(x_0, y_0) + \underbrace{\begin{bmatrix} \frac{\partial f(x_0, y_0)}{\partial x} & \frac{\partial f(x_0, y_0)}{\partial y} \end{bmatrix}}_{1 \times 2 \text{ matrix}} \cdot \underbrace{\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}_{2 \times 1 \text{ matrix}}.$$

## Our Goal

- Our task is to define the linearization at a point  $(x_0, y_0)$  of vector-valued functions whose domain and range are  $\mathbb{R}^2$ ; that is,

$$\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (x, y) \mapsto \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}.$$

- To do so, we linearize at the point  $(x_0, y_0)$  each component of  $\mathbf{h}(x, y)$

$$L_f(x, y) = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$$

$$L_g(x, y) = g(x_0, y_0) + \frac{\partial g(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial g(x_0, y_0)}{\partial y}(y - y_0).$$

- We define the linearization of  $\mathbf{h}(x, y)$  at the point  $(x_0, y_0)$  to be the vector-valued function  $\mathbf{L}(x, y)$

$$\mathbf{L}(x, y) = \begin{bmatrix} L_f(x, y) \\ L_g(x, y) \end{bmatrix}.$$

## The Jacobi (or Derivative) Matrix

We can rewrite the linearization  $\mathbf{L}(x, y)$  at a point  $(x_0, y_0)$  of the vector-valued functions  $\mathbf{h}(x, y)$  in the following matrix form

$$\begin{aligned} \mathbf{h}(x, y) \approx \mathbf{L}(x, y) &= \begin{bmatrix} L_f(x, y) \\ L_g(x, y) \end{bmatrix} \\ &= \begin{bmatrix} f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0) \\ g(x_0, y_0) + \frac{\partial g(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial g(x_0, y_0)}{\partial y}(y - y_0) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}}_{\mathbf{h}(x_0, y_0)} + \underbrace{\begin{bmatrix} \frac{\partial f(x_0, y_0)}{\partial x} & \frac{\partial f(x_0, y_0)}{\partial y} \\ \frac{\partial g(x_0, y_0)}{\partial x} & \frac{\partial g(x_0, y_0)}{\partial y} \end{bmatrix}}_{(D\mathbf{h})(x_0, y_0)} \cdot \begin{bmatrix} (x - x_0) \\ (y - y_0) \end{bmatrix} \end{aligned}$$

$(D\mathbf{h})(x_0, y_0)$  is a  $2 \times 2$  matrix called the **Jacobi matrix** of  $\mathbf{h}$  at  $(x_0, y_0)$ .

### Example 1 (Problem #10, Exam 3, Spring 2012)

Consider the vector valued function  $\mathbf{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$\mathbf{h}(x, y) = \begin{bmatrix} x^2y - y^3 \\ 2x^3y^2 + y \end{bmatrix}.$$

- Compute the **Jacobi matrix**  $(D\mathbf{h})(x, y)$  and evaluate it at the point  $(1, 2)$ .
- Find the linear approximation of  $\mathbf{h}(x, y)$  at the point  $(1, 2)$ .

$$(a) \quad D\mathbf{h}(x, y) = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 - 3y^2 \\ 6x^2y^2 & 4x^3 + 1 \end{bmatrix}$$

$$\text{where } \mathbf{h}(x, y) = \begin{bmatrix} h_1(x, y) \\ h_2(x, y) \end{bmatrix} = \begin{bmatrix} x^2y - y^3 \\ 2x^3y^2 + y \end{bmatrix}$$

$$\text{hence at } (x=1, y=2) \text{ we have } D\mathbf{h}(1, 2) = \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix}$$

$$\begin{aligned} (b) \quad \mathbf{L}(x, y) &= \begin{bmatrix} h_1(1, 2) \\ h_2(1, 2) \end{bmatrix} + D\mathbf{h}(1, 2) \cdot \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ 10 \end{bmatrix} + \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} \\ &= \begin{bmatrix} -6 + 4(x-1) - 11(y-2) \\ 10 + 24(x-1) + 9(y-2) \end{bmatrix} = \begin{bmatrix} 4x - 11y + 12 \\ 24x + 9y - 32 \end{bmatrix} \end{aligned}$$

**Example 2** (Problem #46, Section 10.4, p. 599)

Find a linear approximation to

$$\mathbf{f}(x, y) = \begin{bmatrix} \sqrt{2x+y} \\ x-y^2 \end{bmatrix}$$

at  $(1, 2)$ . Use your result to find an approximation for  $\mathbf{f}(1.05, 2.05)$ .

$$\underline{\mathbf{f}}(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} \sqrt{2x+y} \\ x-y^2 \end{bmatrix}$$

$$\underline{D}\underline{\mathbf{f}}(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2x+y}} & \frac{1}{2\sqrt{2x+y}} \\ 1 & -2y \end{bmatrix}$$

$$\underline{D}\underline{\mathbf{f}}(1, 2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & -4 \end{bmatrix}$$

$$\underline{\mathbf{L}}(x, y) = \underbrace{\begin{bmatrix} 2 \\ -3 \end{bmatrix}}_{\underline{\mathbf{f}}(1, 2)} + \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} = \begin{bmatrix} \frac{x}{2} + \frac{y}{4} + 1 \\ x-4y + 4 \end{bmatrix}$$

$$\underline{\mathbf{f}}(1.05, 2.05) \approx \underline{\mathbf{L}}(1.05, 2.05) = \begin{bmatrix} 2.0375 \\ -3.15 \end{bmatrix}$$

$$\text{exact value} = \begin{bmatrix} 2.03715 \\ -3.1525 \end{bmatrix}$$

**Example 3** (Example # 9, Section 10.4, p. 597)Consider the function  $\mathbf{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$   $(x, y) \mapsto \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$ , with

$$u(x, y) = ye^{-x} \quad \text{and} \quad v(x, y) = \sin x + \cos y.$$

Find the linear approximation to  $\mathbf{f}(x, y)$  at  $(0, 0)$ .Compare  $\mathbf{f}(0.1, -0.1)$  with its linear approximation.

$$\underline{\mathbf{f}}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} ye^{-x} \\ \sin x + \cos y \end{bmatrix}$$

$$\underline{D}\underline{\mathbf{f}}(x, y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} -ye^{-x} & e^{-x} \\ \cos x & -\sin y \end{bmatrix}$$

$$\underline{D}\underline{\mathbf{f}}(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Linear approximation of  $\underline{\mathbf{f}}$  at  $(0, 0)$ 

$$\underline{\mathbf{L}}(x, y) = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\underline{\mathbf{f}}(0, 0)} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x-0 \\ y-0 \end{bmatrix} = \begin{bmatrix} y \\ 1+x \end{bmatrix}$$

$$\underline{\mathbf{f}}(0.1, -0.1) = \begin{bmatrix} -0.09048 \\ 1.0948 \end{bmatrix} \approx \underline{\mathbf{L}}(0.1, -0.1) = \begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix}$$

## The General Case

- Consider the function  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , say  $\mathbf{f}(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{bmatrix}$ ,

where  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , are  $m$  real-valued functions of  $n$  variables.

- The Jacobi matrix of  $\mathbf{f}$  is an  $m \times n$  matrix of the form

$$(D\mathbf{f})(x_1, \dots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

- The linearization of  $\mathbf{f}$  about the point  $(x_1^*, \dots, x_n^*)$  is then

$$\mathbf{L}(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1^*, \dots, x_n^*) \\ \vdots \\ f_m(x_1^*, \dots, x_n^*) \end{bmatrix} + (D\mathbf{f})(x_1^*, \dots, x_n^*) \cdot \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}$$