Direction fields of differential equations...with SAGE

Many differential equations cannot be solved conveniently by analytical methods, so it is important to consider what qualitative information can be obtained about their solutions without actually solving the equations.

A **direction field** (or **slope field**) is a graphical representation of the solutions of a first-order differential equation of the form

$$\frac{dy}{dx} = f(x, y).$$

Imagine that you are standing at a point $P(\alpha, \beta)$ in the xy-plane and that the above differential equation determines your future location. Where should you go next? You move along a curve whose tangent line at the point $P(\alpha, \beta)$ has slope $dy/dx|_P = f(\alpha, \beta)$.

We (or, better, a computer) can construct a direction field (or slope field) by evaluating the function $f(x, y)$ at each point of a rectangular grid consisting of at least a few hundred points. Then, at each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a given differential equation.

The graph of a solution to the given differential equation is a curve in the xy-plane. It is often useful to regard this curve as the path, or **trajectory** traversed by a moving particle. The xy-plane is called the **phase plane** and a representative set of trajectories is referred to as a **phase portrait**.

SAGE is a free open-source mathematics software system. You can download this software or use it online at the following address

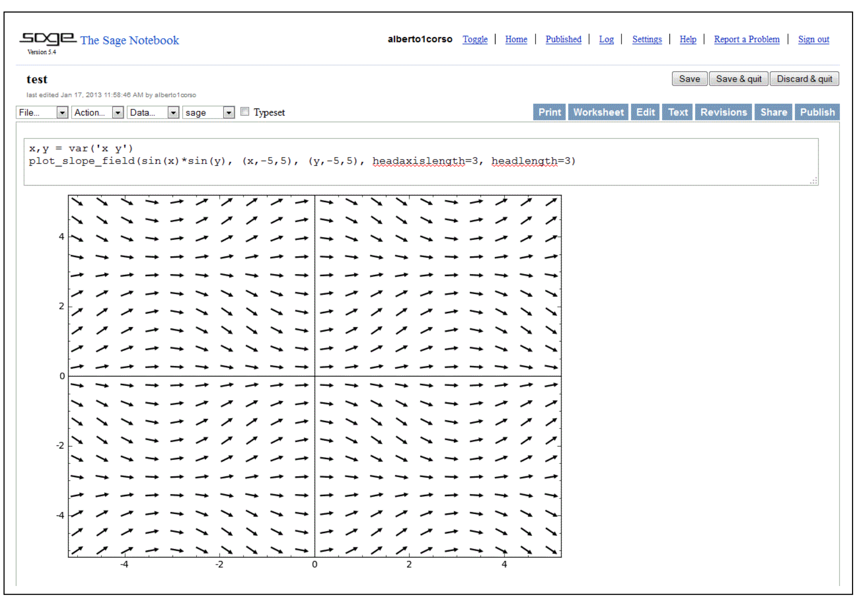
`www.sagemath.org/`

To try sage online follow the appropriate links at the above address and select one of the OpenID providers (say, for example, Google or Yahoo).

It is easy to plot direction (slope) fields of a differential equation using SAGE. For this we use the command

`plot_slope_field`

The picture below shows you a snapshot of a session in SAGE with the direction field of the differential equation $dy/dx = \sin(x)\sin(y)$.



Example 1: Consider the differential equation

$$\frac{dy}{dx} = x^2y^2 \qquad \rightsquigarrow \qquad \int \frac{dy}{y^2} = \int x^2\,dx.$$

It is easy to check that the general solution to this differential equation is given by the function

$$y = \frac{-3}{x^3 + C},$$

where C is a constant. If we make the constant equal to $6, -3$, and 0.3, respectively, we obtain the three solutions below

$$y_1 = \frac{-3}{x^3+6} \qquad y_2 = \frac{-3}{x^3-3} \qquad y_3 = \frac{-3}{x^3+0.3}$$

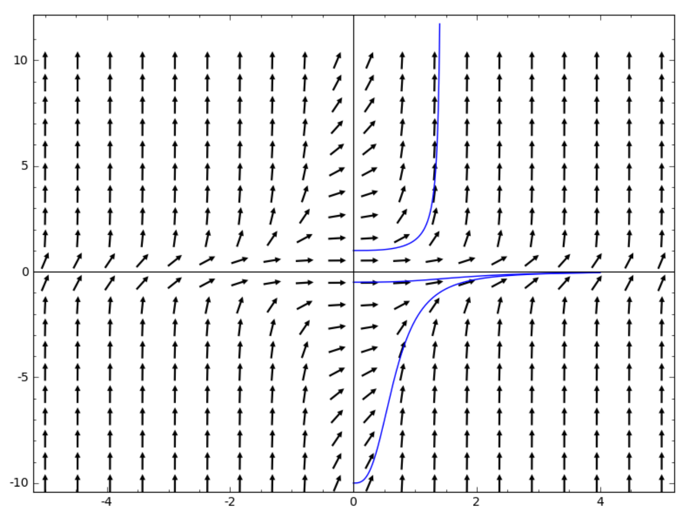
which correspond to the initial conditions

$$y_1(0) = -0.5 \qquad y_2(0) = 1 \qquad y_3(0) = -10$$

respectively.

Below are the commands to plot the direction field of the given differential equation as well as the graphs of those three particular solutions.

```
x,y=var('x,y')
v=plot_slope_field(x^2*y^2,(x,-5,5),(y,-10,10),headaxislength=3, headlength=3)
a=6
b=-3
c=0.3
d1=plot(-3/(x^3+a),(x,0,4))
d2=plot(-3/(x^3+b),(x,0,1.4))
d3=plot(-3/(x^3+c),(x,0,4))
show(v+d1+d2+d3)
```



Phase Portrait 1: direction field of $dy/dy = x^2y^2$ and some particular solutions.

Example 2: Consider the differential equation

$$\frac{dy}{dx} = y^2 - 4 \qquad \rightsquigarrow \qquad \int \frac{dy}{(y-2)(y+2)} = \int dx.$$

Using the method of partial fractions, integration and a few algebraic manipulations, we see that the general solution to this differential equation is given by the function

$$y = 2 \cdot \frac{1 + Ce^{4x}}{1 - Ce^{4x}} = 2 \cdot \frac{e^{-4x} + C}{e^{-4x} - C},$$

where C is a constant. If we make the constant equal to $2, -1$, and 0.1, respectively, we obtain the three solutions

$$y_1 = 2 \cdot \frac{1 + 2e^{4x}}{1 - 2e^{4x}} \qquad y_2 = 2 \cdot \frac{1 - e^{4x}}{1 + e^{4x}} \qquad y_3 = 2 \cdot \frac{1 + 0.1e^{4x}}{1 - 0.1e^{4x}},$$

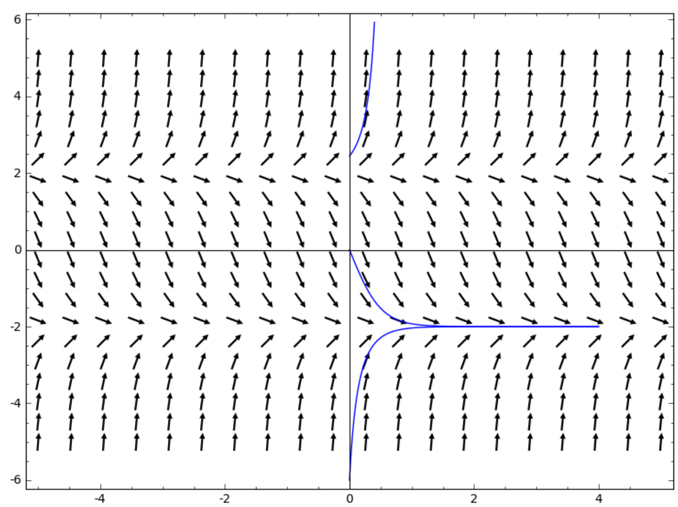
which correspond to the initial conditions

$$y_1(0) = -6 \qquad y_2(0) = 0 \qquad y_3(0) = \frac{22}{9} \approx 2.\overline{4},$$

respectively.

The commands to plot the direction field of the given differential equation as well as the graphs of those three particular solutions are below. Please notice the long term behavior of those three solutions!

```
x,y=var(’x,y’)
v=plot_slope_field((y^2-4),(x,-5,5),(y,-5,5),headaxislength=3, headlength=3)
a=2
b=-1
c=0.1
d1=plot(2*(1+a*e^(4*x))/(1-a*e^(4*x)),(x,0,4))
d2=plot(2*(1+b*e^(4*x))/(1-b*e^(4*x)),(x,0,4))
d3=plot(2*(1+c*e^(4*x))/(1-c*e^(4*x)),(x,0,0.4))
show(v+d1+d2+d3)
```



Phase Portrait 2: direction field of $dy/dy = y^2 - 4$ and some particular solutions.

Remark: If you compute the limit as x tends to infinity of $\quad y = 2 \cdot \dfrac{1 + Ce^{4x}}{1 - Ce^{4x}} = 2 \cdot \dfrac{e^{-4x} + C}{e^{-4x} - C}$ you see that for any choice of C the limit is -2. This seems inconsistent with the behavior of y_3 in the phase portrait above. (It seems very different from the behavior of y_1 and y_2.) This difference is due to the fact that

$$\lim_{x \to \ln(10)/4} 2 \cdot \frac{1 + 0.1e^{4x}}{1 - 0.1e^{4x}} = +\infty,$$

that is, the solution y_3 has a discontinuity at $x = \ln(10)/4$.

Example 3 (Logistic growth model): A biological population with plenty of food, space to grow, and no threat from predators, tends to grow at a rate that is proportional to the population – that is, in each unit of time, a certain percentage of the individuals produce new individuals. If reproduction takes place more or less continuously, then this growth rate is represented by

$$\frac{dN}{dt} = rN,$$

where $N = N(t)$ is the population as a function of time t, and r is the proportionality constant. We know that all solutions of this natural-growth equation have the form

$$N(t) = N_0 e^{rt},$$

where N_0 is the population at time $t = 0$.

In short, unconstrained natural growth is exponential growth. However, we may account for the growth rate declining to 0 by including a factor $1 - N/K$ in the model, where K is a positive constant. The factor $1 - N/K$ is close to 1 (that is, has no effect) when N is much smaller than K, and is close to 0 when N is close to K. The resulting model,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad \text{with} \qquad N(0) = N_0$$

is called the **logistic growth model** or the **Verhulst model**[1].

To obtain the solution to this differential equation we proceed as follows:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \quad \rightsquigarrow \quad \frac{1}{N\left(1 - \dfrac{N}{K}\right)}\, dN = r\, dt \quad \rightsquigarrow \quad \frac{K}{N(K-N)}\, dN = r\, dt$$

Using the method of partial fractions, integration and a few algebraic manipulations, we obtain that the general solution to this differential equation is given by the function

$$\int\left(\frac{1}{N} + \frac{1}{K-N}\right) dN = \int r\, dt \quad \rightsquigarrow \quad \ln(N) - \ln(K-N) = rt + C \quad \rightsquigarrow \quad \frac{N(t)}{K - N(t)} = Ae^{rt},$$

where C and $A = e^C$ are constants. To determine the value of the constant A we now use the initial condition $N(0) = N_0$. We find that $A = N_0/(K - N_0)$. Thus our solution (after a few algebraic manipulations) looks like

$$\frac{N(t)}{K - N(t)} = \frac{N_0}{K - N_0} e^{rt} \quad \rightsquigarrow \quad \frac{K - N(t)}{N(t)} = \frac{K - N_0}{N_0 e^{rt}} \quad \rightsquigarrow \quad K - N(t) = N(t)\left(\frac{K}{N_0} - 1\right)e^{-rt}$$

$$\rightsquigarrow \quad \boxed{\boxed{N(t) = \frac{K}{1 + \left(\dfrac{K}{N_0} - 1\right)e^{-rt}}}}$$

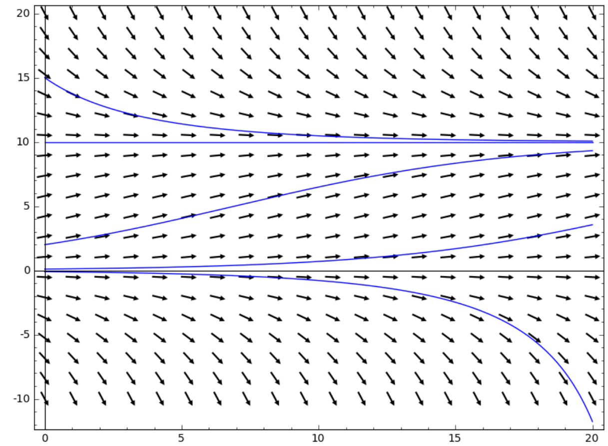
Observe that $\lim_{t\to\infty} N(t) = K$. This justifies the fact that the constant K is dubbed **carrying capacity**.

Here is a numerical example with $r = 0.2$ (that is we assume a 20% growth rate) and $K = 10$.

$$\frac{dN}{dt} = 0.2\, N\,(1 - N/10) \quad \rightsquigarrow \quad N(t) = \frac{10}{1 + (10/N_0 - 1)e^{-0.2t}}$$

[1] The word "logistic" has no particular meaning in this context, except that it is commonly accepted. The second name honors Pierre François Verhulst (1804–1849), a Belgian mathematician who studied this idea in the 19th century. Using data from the first five U.S. censuses, he made a prediction in 1840 of the U.S. population in 1940 – and was off by less than 1%.

The direction field of the given differential equation as well as the graphs of particular solutions are below. Please notice the long term behavior of those solutions!

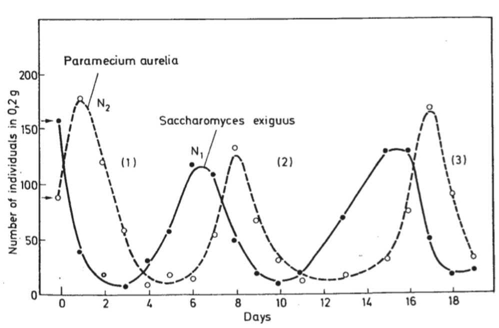


Phase Portrait 3: direction field of $dN/dt = 0.2\,N\,(1 - N/10)$ and some particular solutions.

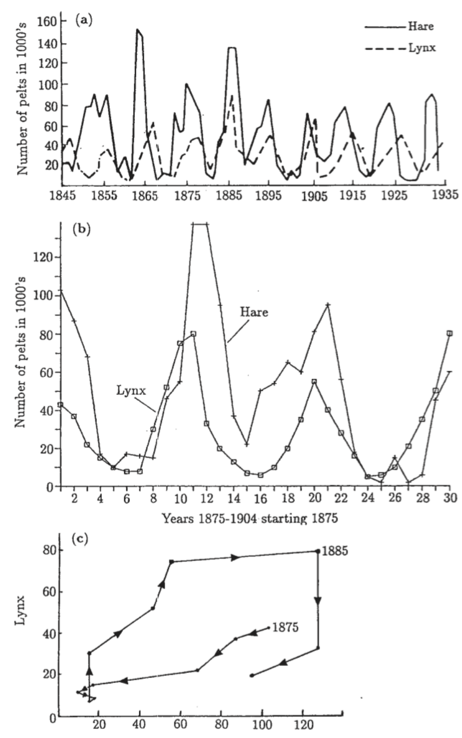
Example 4 (Lotka-Volterra predator-prey model): We give an example of a class of differential equations that describes the interaction of two species in a way in which one species (**the predator**) preys on the other species (**the prey**), while the prey lives on a different source of food. The population distributions tend to show periodic oscillations. We stress upfront that a model involving only two species cannot fully describe the complex relationship among species that actually occur in nature. Nevertheless, the study of simple models is the first step toward an understanding of more complicated phenomena.

When the prey population increases in size, the predatory species obtains a larger food base. Hence, with a certain time delay it will also become more numerous. As a consequence, the growing pressure for food will reduce the prey population. After a while food becomes rare for the predator species so that its propagation is inhibited. The size of the predator population will decline. The new phase favors the prey population. Slowly it will grow again, and the pattern in changing population sizes may repeat. When conditions remain the same, the process continues in cycles.

The figures on the side and below illustrate such a cyclical dynamics.



Legend: Fluctuation of population size of *Paramecium aurelia* which feeds upon *Saccharomyces exiguus*.



Legend:

(a) Fluctuations in the number of pelts sold by the Hudson Bay Company.

(b) Detail of the 30-year period starting in 1875.

(c) Phase plane plot of the data in **(b)**.

A (highly simplified) model for the predator-prey interaction can be summarized as follows:

$$\left\{\begin{array}{l}\text{change in}\\\text{the number}\\\text{of prey}\end{array}\right\} = \left\{\begin{array}{l}\text{natural}\\\text{increase}\\\text{in prey}\end{array}\right\} - \left\{\begin{array}{l}\text{destruction}\\\text{of prey by}\\\text{predator}\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{change in}\\\text{the number}\\\text{of predator}\end{array}\right\} = \left\{\begin{array}{l}\text{increase in}\\\text{predator resulting}\\\text{from devouring prey}\end{array}\right\} - \left\{\begin{array}{l}\text{natural}\\\text{loss in}\\\text{predator}\end{array}\right\}$$

We now translate this model into differential equations. Let $x = x(t)$ be the number of prey individuals and $y = y(t)$ the number of predator individuals at time instant t. We assume that x and y are differentiable functions of t.

The **key assumptions** in the Lotka-Volterra[2,3] model are:

- the birth rate of the prey species is likely to be proportional to x, that is, equal to ax with a certain constant $a > 0$;
- the destruction rate depends on x and on y. The more prey individuals are available, the easier it is to catch them, and the more predator individuals are around, the more stomachs have to be fed. It is reasonable to assume that the destruction rate is proportional to x and to y, that is, equal to bxy with a certain constant $b > 0$.
- the birth rate of the predator population depends on food supply as well as on its present size. We may assume that the birth rate is proportional to x and to y, that is, equal to cxy with a certain constant $c > 0$.
- the death rate of the predator species is likely to be proportional to y, that is, equal to dy with a certain $d > 0$.

Under these simplifying assumptions the differential equations that we obtain are:

$$\frac{dx}{dt} = ax - bxy \qquad \frac{dy}{dt} = cxy - dy.$$

How do we deal these equations? Because of the interaction between the two populations x (prey) and y (predator), we can view y as a function of x. As a consequence of the chain rule, we have

$$\underbrace{\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}}_{\text{chain rule}} \quad \rightsquigarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \rightsquigarrow \quad \boxed{\boxed{\frac{dy}{dx} = \frac{(cx-d)y}{x(a-by)}.}}$$

Here is a numerical example with $a = 1$, $b = 4$, $c = 2$, and $d = 3$, so that

$$\boxed{\boxed{\frac{dy}{dx} = \frac{(2x-3)y}{x(1-4y)}.}}$$

[2] Alfred Lotka (March 2, 1880–December 5, 1949) was a Polish-born mathematician, physical chemist, and statistician, best known for his proposal of the predator-prey model, developed simultaneously but independently of Vito Volterra. The Lotka-Volterra model is still the basis of many models used in the analysis of population dynamics in ecology.

[3] Vito Volterra (3 May, 1860–11 October, 1940) was an Italian mathematician and physicist, known for his contributions to mathematical biology and integral equations.

If we separate the variables this leads to

$$\frac{1-4y}{y}dy = \frac{2x-3}{x}dx \quad \rightsquigarrow \quad (1/y-4)\,dy = (2-3/x)\,dx.$$

After integrating we obtain the solution

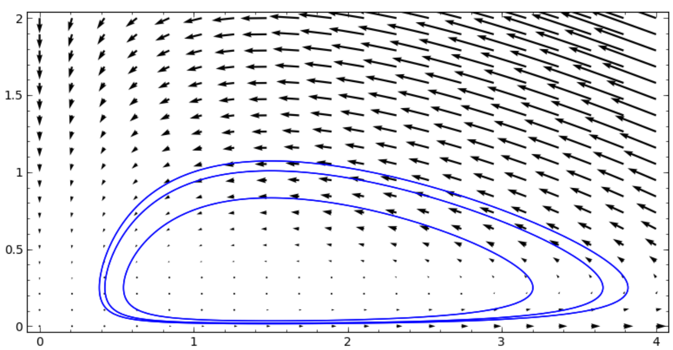
$$\ln y - 4y = 2x - 3\ln x + C \quad \rightsquigarrow \quad \ln y + \ln(e^{-4y}) + \ln(x^3) + \ln(e^{-2x}) = C \quad \rightsquigarrow$$

$$\boxed{\boxed{ye^{-4y}x^3e^{-2x} = \kappa,}}$$

where C and $\kappa = e^C$ are constants.

It is worth mentioning that we can write the general solution of the arbitrary Lotka-Volterra equation in the same fashion.

The direction field of the differential equation $dy/dx = \dfrac{(2x-3)y}{x(1-4y)}$ is different from the ones we saw earlier. Again it has been produced with the SAGE commands introduced earlier.



Phase Portrait 4: direction field of $dy/dx = \dfrac{(2x-3)y}{x(1-4y)}$ and some particular solutions.

Notice that the trajectories are closed curves. Furthermore, they all seem to evolve around the point $P(3/2, 1/4)$. This is the point where the factors $2x-3$ and $1-4y$ of dy/dt and dx/dt, respectively, are both zero.

This confirms our heuristics that the two populations should exhibit a cyclic dynamic.

Example 5 (Solow's economic growth model/Von Bertalanffy's individual growth model): These two models are two different reincarnations of the same differential equation, namely

$$\frac{dy}{dx} = ay^m - by^n,$$

where $a, b, m,$ and n are constants.

- **Solow's economic growth model:** The capital stock $k = k(t)$ varies over time t, increasing as a result of investments and decreasing as a result of depreciation. With these basic assumptions and using a Cobb-Douglass production function, the Solow's growth economic model[4] becomes

$$\frac{dk}{dt} = sk^\alpha - \delta k \qquad \text{with} \qquad k(0) = k_0,$$

  where s, α, δ are constants $0 < s, \alpha < 1$ and $\delta > 0$. The constants s and δ are called the rate of savings and the depreciation rate, respectively.

[4]Robert Solow (born August 23, 1924) is an American economist particularly known for his work on the theory of economic growth that culminated in the exogenous growth model named after him. He was awarded the John Bates Clark Medal (in 1961) and the 1987 Nobel Prize in Economics.

- **Von Bertalanffy individual growth model:** The individual growth model published by von Bertalanffy[5] in 1934 is widely used in biological models and exists in a number of permutations. In one of its forms it says that the change of body weight W of an individual is given by the difference between the process of building up (anabolism) and breaking down (catabolism)
$$\frac{dW}{dt} = \eta W^{2/3} - \kappa W \qquad \text{with} \qquad W(0) = W_0,$$
where η and κ are the constants of anabolism and catabolism, respectively. The exponents 2/3 and 1 indicate that the latter (anabolism and catabolism) are proportional to some powers of the body weight W.

  An even simpler type of von Bertalanffy growth equation that we have encountered so far says that the length $L = L(t)$ over time t of a fish is given by:
$$\frac{dL}{dt} = r_B\,(L_\infty - L) \qquad \text{with} \qquad L(0) = L_0,$$
where r_B is the von Bertalanffy growth rate and L_∞ the ultimate length of the fish. In this case the powers of L that appear in the differential equation are 0 and 1 respectively. We have also seen that the solution in this case is $L(t) = L_\infty - (L_\infty - L_0)e^{-r_B t}$.

Consider the differential equation given earlier in the case $n = 1$, that is
$$\boxed{\boxed{\frac{dy}{dx} = a\,y^m - b\,y}} \qquad \rightsquigarrow \qquad \frac{dy}{dx} = y^m(a - b\,y^{1-m}).$$
This suggests the use of the substitution $u = y^{1-m}$. The chain rule says that
$$\frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx} \qquad \rightsquigarrow \qquad \frac{du}{dx} = (1-m)\,y^{[(1-m)-1]}\,\frac{dy}{dx} \qquad \rightsquigarrow \qquad \frac{1}{1-m}\,y^m\,\frac{du}{dx} = \frac{dy}{dx}.$$
If we now substitute the latter expression into our original differential equation we get the separable differential equation below
$$\frac{du}{dx} = (1-m)\,(a - b\,u).$$
We now separate the variables, multiply both sides by $-b$, and then integrate. We obtain
$$\frac{b}{b\,u - a}\,du = -(1-m)\,b\,dx \qquad \rightsquigarrow \qquad \ln(b\,u - a) = -(1-m)\,b\,x + C$$
where C is a constant. After a few additional simple algebraic manipulations and after substituting back y^{1-m} in place of u, we obtain
$$y = \left[\frac{a}{b} + D\cdot e^{-(1-m)\,b\,x}\right]^{1/(1-m)},$$
where $D = e^C/b$. The initial condition $y(0) = y_0$ gives us the following value for the constant D: namely, $D = y_0^{1-m} - \dfrac{a}{b}$. Thus the solution of our initial value problem is
$$\boxed{\boxed{y = \left[\frac{a}{b} - \left(\frac{a}{b} - y_0^{1-m}\right)\cdot e^{-(1-m)\,b\,x}\right]^{1/(1-m)}.}}$$

[5] Karl Ludwig von Bertalanffy (September 19, 1901, Atzgersdorf near Vienna–June 12, 1972, Buffalo, New York) was an Austrian-born biologist known as one of the founders of general systems theory (GST). GST is an interdisciplinary practice that describes systems with interacting components, applicable to biology, cybernetics, and other fields. Bertalanffy proposed that the laws of thermodynamics applied to closed systems, but not necessarily to "open systems," such as living things. His mathematical model of an organism's growth over time, published in 1934, is still in use today.

Notice that $y_\infty = \lim_{x\to\infty} y = \left[\frac{a}{b}\right]^{1/(1-m)}$.

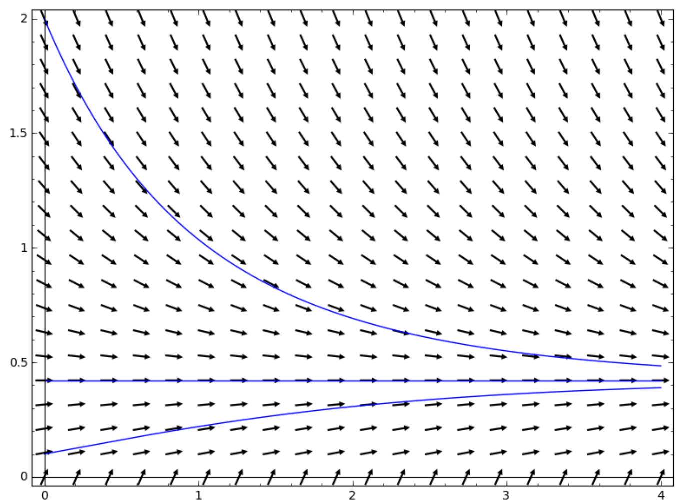
In the special case when $m = 2/3$, our model with initial condition $y(0) = y_0$ and asymptotic value y_∞ simplifies to

$$\frac{dy}{dx} = a\,y^{2/3} - b\,y \qquad \rightsquigarrow \qquad y = \left[\frac{a}{b} - \left(\frac{a}{b} - y_0^{1/3}\right)\cdot e^{-b\,x/3}\right]^3 \qquad y_\infty = \left[\frac{a}{b}\right]^3.$$

Here is a numerical example with $a = 1.5$, $b = 2$, $m = 2/3$, and $n = 1$, so that

$$\frac{dy}{dx} = 1.5\,y^{2/3} - 2\,y \qquad \rightsquigarrow \qquad y = \left[0.75 - \left(0.75 - \sqrt[3]{y_0}\right)e^{-2/3x}\right]^3.$$

The direction field of the given differential equation as well as the graphs of particular solutions are below. Please notice the long term behavior of those solutions! As x approaches infinity the solution approaches the value $y_\infty = (1.5/2)^3 \approx 0.422$.



Phase Portrait 5: direction field of $dy/dx = 1.5\,y^{2/3} - 2\,y$ and some particular solutions.

References

[1] E. Batschelet, *Introduction to Mathematics for Life Scientists.*

[2] L. Edelstein-Keshet, *Mathematical Models in Biology.*

[3] D. E. Goldberg and J. H. Vandermeer, *Population Ecology.*

[4] J.D. Murray, *Mathematical Biology.*

[5] C. Neuhauser, *Calculus for Biology and Medicine.*

[6] C. H. Taubes, *Modeling Differential Equations in Biology.*