$\boxed{\text{WORKSHEET \#10}}$

$\boxed{\#1}$ $\quad \dfrac{dy}{dx} = (x-2)e^{-2y}$ with $y(2) = \ln(2)$

$$\Longleftrightarrow \int \frac{1}{e^{-2y}}\,dy = \int (x-2)\,dx$$

$$\Longleftrightarrow \int e^{2y}\,dy = \int (x-2)\,dx$$

$$\frac{1}{2}e^{2y} = \frac{1}{2}x^2 - 2x + C$$

get rid of 2 and relabel the constant

$$e^{2y} = x^2 - 4x + \tilde{C} \qquad [\tilde{C} = 2C]$$

To get y Take the natural log of both sides. However before we do so, let's get the value of $\tilde{C}$:

$$\underbrace{e^{2(\ln 2)}}_{= e^{\ln(2^2)}} = 2^2 - 4\cdot 2 + \tilde{C} \qquad [y(2) = \ln(2)]$$

$$= 4 - 8 + \tilde{C}$$

$$\cancel{4} = \cancel{4} - 8 + \tilde{C} \qquad \boxed{\tilde{C} = 8}$$

So $\quad e^{2y} = x^2 - 4x + 8 \quad$ and

$$\ln\left(e^{2y}\right) = \ln\left(x^2-4x+8\right)$$

$$\Longleftrightarrow \quad 2y = \ln\left(x^2-4x+8\right)$$

$$\Longleftrightarrow \quad y = \frac{1}{2}\ln\left(x^2-4x+8\right) = \ln\left[\sqrt{x^2-4x+8}\right]$$

using the properties of logarithms

$\boxed{\#2}$ (a) $\dfrac{dy}{dx} = e^{-3x}$ $\qquad y_0 = 10$ for $x_0 = 0$

$$\Longleftrightarrow \quad y = 10 + \int_0^x e^{-3u}\,du$$

$$= 10 + \left(-\frac{1}{3}e^{-3u}\Big|_0^x\right)$$

$$= 10 + \left(-\frac{1}{3}e^{-3x} - \left(-\frac{1}{3}e^0\right)\right)$$

$$= 10 + \left(\frac{1}{3} - \frac{1}{3}e^{-3x}\right) = \boxed{\frac{31}{3} - \frac{1}{3}e^{-3x}}$$

Alternatively, $\displaystyle\int dy = \int e^{-3x}\,dx$

$$\Longleftrightarrow \quad y = -\frac{1}{3}e^{-3x} + C$$

Use the initial condition

$$10 = -\frac{1}{3} e^{-3(0)} + C \quad \text{so}$$

$$10 + \frac{1}{3} = C \iff C = \frac{31}{3}$$

Thus $\quad y = \frac{31}{3} - \frac{1}{3} e^{-3x}$ as before.

(b) $\quad \frac{dx}{dt} = \frac{1}{1-t} \qquad x(0) = 2$

derivative of denom

$$\iff x = \int dx = \int \frac{1}{1-t} \, dt = -\int \frac{-1}{1-t} \, dt$$

$$= -\ln|1-t| + C$$

So $\quad x = -\ln|1-t| + C$. Now $x(0) = 2$

so $\quad 2 = -\underbrace{\cancel{\ln|1|}}_{0} + C \qquad \therefore C = 2$

$$\boxed{x = x(t) = 2 - \ln|1-t|}$$

Alternatively: $\frac{dx}{dt} = \frac{1}{1-t} \quad x(0) = 2$ has

solution $\quad x = 2 + \int_0^t \frac{1}{1-u} \, du = \dots = \underline{2 - \ln|1-t|}$

(c) $\dfrac{ds}{dt} = \sqrt{3t+1} \qquad s(0)=1$

$$s(t) = \underbrace{1}_{s(0)} + \int_0^t \sqrt{3u+1}\, du$$

use a substitution $w = 3u+1$ so $dw = 3\,du$

or $\frac{1}{3} dw = du$. Thus

$$s(t) = 1 + \int_{3\cdot 0+1}^{3\cdot t+1} \sqrt{w} \cdot \frac{1}{3} dw = 1 + \int_1^{3t+1} \frac{1}{3} w^{1/2} dw$$

$$= 1 + \left(\frac{1}{3}\cdot\frac{2}{3} w^{3/2}\Big|_1^{3t+1}\right) = 1 + \left(\frac{2}{9}(3t+1)^{3/2} - \frac{2}{9}\right)$$

$$\boxed{= \frac{7}{9} + \frac{2}{9}(3t+1)^{3/2}}$$

$\boxed{\#3}$ (a) $\dfrac{dx}{dt} = -2x \qquad x(1) = 5$

$$\Longleftrightarrow \int \frac{1}{x}\, dx = \int -2\, dt$$

$$\Longleftrightarrow \ln|x| = -2t + C$$

take exponential of both sides

$$e^{\ln|x|} = e^{-2t+C}$$

$$\Longleftrightarrow \quad |x| = e^{C} \cdot e^{-2t} \qquad \text{(properties of exponents)}$$

$$\Longleftrightarrow \quad x = \underbrace{\pm e^{C}}_{A} \cdot e^{-2t} \qquad \text{(getting rid of } |\cdot| \text{)}$$

rename constant

$$x = A \cdot e^{-2t}$$

use $x(1) = 5 \longrightarrow 5 = A \cdot e^{-2}$

So $A = \frac{5}{e^{-2}} = \underline{\underline{5 \cdot e^{2}}}$

Hence $x = x(t) = 5e^{2} \cdot e^{-2t} = \boxed{5\, e^{-2(t-1)}}$

(b) $\frac{dh}{dt} = 2h + 1 \qquad h(0) = 4$

$$\Longleftrightarrow \int \frac{1}{2h+1}\, dh = \int 1 \cdot dt$$

Multiply by 2 both sides [2 = derivative of $2h+1$ wrt h]

$$\Longleftrightarrow \int \frac{2}{2h+1}\, dh = \int 2\, dt$$

$\ln|2h+1| = 2t + C$

Take exponential to solve for h:

$$e^{\ln|2h+1|} = e^{2t+C}$$

$$|2h+1| = e^{C} \cdot e^{2t}$$

$$2h+1 = \underbrace{\pm e^{C}}_{A} \cdot e^{2t}$$

we got rid of $|\cdot|$ and renamed the constant

Now when $t=0$ we have $h=4$ by assumption

$2 \cdot 4 + 1 = A \cdot e^{0}$ so $A = 9$

Thus $2h+1 = 9 \cdot e^{2t}$

or $\boxed{h = h(t) = -\tfrac{1}{2} + \tfrac{9}{2} e^{2t}}$

$= \tfrac{1}{2}(9e^{2t} - 1)$

(c) $\dfrac{dy}{dx} = 2y(3-y)$ $y(0) = -3$

Separate variables:

$$\int \frac{1}{y(3-y)}\,dy = \int 2\,dx$$

use partial fractions

$$\frac{1}{y(3-y)} = \frac{A}{y} + \frac{B}{3-y} = \frac{A(3-y) + B\cdot y}{y(3-y)}$$

Thus we need $1 = A(3-y) + B\cdot y$

set $y=0$ to get $1 = A\cdot 3 + B\cdot 0$ so $A = 1/3$

set $y=+3$ to get $1 = A\cdot 0 + B\cdot 3$ So $B = 1/3$

Thus $\int \left(\frac{1/3}{y} + \frac{1/3}{3-y}\right) dy = \int 2\,dx$

Multiply all sides by 3. so we get rid of $1/3$

$$\int \left(\frac{1}{y} + \frac{1}{3-y}\right) dy = \int 6\,dx$$

$$\int \left(\frac{1}{y} - \frac{1}{y-3}\right) dy = \int 6\,dx$$

$$\ln|y| - \ln|y-3| = 6x + C$$

<u>OR</u> $\ln\left|\frac{y}{y-3}\right| = 6x + C$

We get rid of ln and $|\cdot|$

$$e^{\ln|y/y-3|} = e^{6x+C}$$

$$\left|y/y-3\right| = e^C \cdot e^{6x}$$

$$\frac{y}{y-3} = \underbrace{\pm e^C}_{A} \cdot e^{6x}$$

$$\frac{y}{y-3} = A \cdot e^{6x}$$

The initial condition is $y(0) = -3$

so $\frac{-3}{-6} = A \cdot e^0$. Thus $A = 1/2$

Hence $\frac{y}{y-3} = \frac{1}{2} e^{6x}$ Solve for y

$$2y = (y-3)e^{6x} \iff 2y = ye^{6x} - 3e^{6x}$$

$$2y - ye^{6x} = -3e^{6x} \iff y(2 - e^{6x}) = -3e^{6x}$$

$$\boxed{y = \frac{-3e^{6x}}{2 - e^{6x}} = \frac{-3}{2e^{-6x} - 1}}$$

Multiply top and bottom by "e^{-6x}"