WORKSHEET #11

#1 $\dfrac{dN}{dt} = \dfrac{1}{100} N^2 \qquad N(0) = 10$

separate variables $\displaystyle\int \frac{1}{N^2}\, dN = \int \frac{1}{100}\, dt$

$$\int N^{-2}\, dN = \int \frac{1}{100}\, dt \iff -\frac{1}{N} = \frac{1}{100} t + C$$

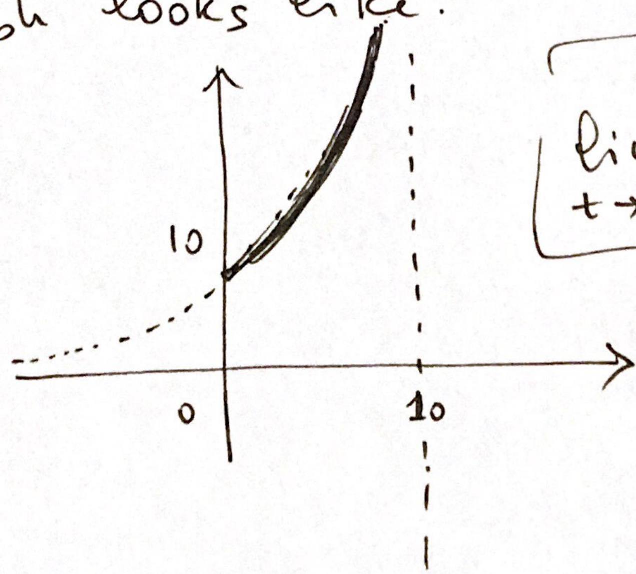
As $N(0)=10$ we have $-\dfrac{1}{10} = \cancel{\dfrac{1}{100} \cdot 0} + C$ $\quad 0$

So $C = -1/10$ and so

$$-\frac{1}{N} = t/100 - \frac{1}{10} \iff \frac{1}{N} = \frac{1}{10} - t/100$$

OR $\dfrac{1}{N} = \dfrac{10-t}{100} \iff \boxed{\dfrac{100}{10-t} = N(t)}$

The Graph looks like:



$$\boxed{\lim_{t \to 10^-} \frac{100}{10-t} = +\infty}$$

$\boxed{\#2}$ $\frac{dL}{dt} = k\,(123 - L)$ with $L(0)=1$

$\Longleftrightarrow \int \frac{1}{123-L}\,dL = \int k\,dt$

need to multiply by "-1" on both sides

$$\int \frac{-1}{123-L}\,dL = \int -k\,dt$$

so $\ln(\underbrace{123-L}_{\text{quantity is } \geq 0}) = -kt + C$

Take exponential

$123 - L = \underbrace{e^{C}}_{A} \cdot e^{-kt}$ so using $L(0)=1$

we get $123 - 1 = A \cdot \underbrace{e^{0}}_{=1}$ so $A = 122$

$$\boxed{L(t) = 123 - 122\,e^{-kt}}$$

We need to find k, Knowing that

$L(27) = \frac{1}{2}(123)$

So $\frac{1}{2}\,123 = 123 - 122\,e^{-k\cdot(27)}$

So $\quad 122\, e^{-27k} = 123 - \frac{1}{2}\cdot 123$

$$122\, e^{-27k} = \frac{1}{2}\cdot 123$$

OR $\quad e^{-27k} = \frac{1}{2}\cdot\frac{123}{122}$

So that $\quad -27k = \ln\left(\frac{1}{2}\cdot\frac{123}{122}\right)$

OR $\quad k = -\frac{1}{27}\cdot\ln\left(\frac{1}{2}\cdot\frac{123}{122}\right) \approx 0.02537$

and $\quad \boxed{L(t) = 123 - 122\, e^{-0.02537t}}$

We are asked to compute $L(10) = 123 - 122\, e^{-0.2537}$

≈ 37.337 inches

Finally we are asked to find "t" such that

$L(t) = 0.9 \times 123 = 110.7$ inches

$$110.7 = 123 - 122\, e^{-0.02537t}$$

$$\Leftrightarrow \quad 122\, e^{-0.02537t} = 123 - 110.7$$

$$\Leftrightarrow \quad e^{-0.02537t} = \frac{123-110.7}{122}$$

OR

$$t = -\frac{1}{0.02537}\ln\left(\frac{123-110.7}{122}\right)$$

≈ 90.438

#3. $\dfrac{dy}{dt} = k(900{,}000 - y)$ for some $k > 0$

$\dfrac{1}{900{,}000 - y}\,dy = k\,dt \iff \displaystyle\int \frac{-1}{900{,}000-y}\,dy = -\int k\,dt$

$\iff \ln(900{,}000 - y) = -kt + C$

↳ this quantity is ≥ 0

$900{,}000 - y = \underbrace{e^{C}}_{A \text{ a constant}} \cdot e^{-kt}$

Now $y(0) = 0$ (no one has heard the news at first)

$\iff 900{,}000 = A \cdot \underbrace{e^{0}}_{1}$ $\therefore$ $900{,}000 = A$

Hence $\boxed{y(t) = 900{,}000 - 900{,}000\,e^{-kt}}$

We know that $y(6) = 450{,}000$ so we can obtain $\underline{\underline{k}}$.

$900{,}000 - 900{,}000\,e^{-6k} = 450{,}000$

$\iff 450{,}000 = 900{,}000\,e^{-6k}$

OR $\quad +\frac{1}{2} = e^{-6k}$ Take ln of both sides

$\ln(\frac{1}{2}) = -6k$ OR $k = \frac{\ln(2)}{6}$

≈ 0.89588

$\boxed{\#4.}$

$$\frac{dP}{dt} = k \cdot P\left(1 - \frac{P}{9{,}500}\right) \iff$$

$$\frac{1}{P\left(1 - \frac{P}{9{,}500}\right)} dP = k\,dt \iff$$

$$\int \frac{9{,}500}{P(9{,}500 - P)} dP = \int k\,dt$$

We need to use partial fractions.
Check that

$$\frac{9{,}500}{P(9{,}500 - P)} = \frac{1}{P} + \frac{1}{9{,}500 - P} \quad \text{so}$$

$$\int \left(\frac{1}{P} + \frac{1}{9{,}500 - P}\right) dP = \int k\,dt$$

$$\int \left(\frac{1}{P} - \frac{-1}{9{,}500 - P}\right) dP = kt + C$$

$$\ln P - \ln(9{,}500 - P) = kt + C$$

$$\iff \ln\left(\frac{P}{9{,}500-P}\right) = kt + C$$

OR $$\frac{P}{9{,}500 - P} = \underbrace{e^{C}}_{A} \cdot e^{kt}$$

We know $P(0) = \underline{\underline{500}}$ So

$$\frac{500}{9{,}500 - 500} = A \cdot \underbrace{e^{0}}_{1} \quad \text{so} \quad \boxed{A = 5/90} = \frac{1}{18}$$

$$\frac{P}{9{,}500-P} = 5/90 \cdot e^{kt} \iff \frac{9{,}500-P}{P} = \frac{90}{5e^{kt}}$$

OR $$9{,}500 - P = 90/5 \cdot P \cdot e^{-kt}$$

OR $$9{,}500 = 90/5\, P e^{-kt} + P$$

OR $$P(t) = \frac{9{,}500}{90/5\, e^{-kt} + 1}$$

$P(1) = 1{,}500$ allows us to find k.

$$1{,}500 = \frac{9{,}500}{90/5\, e^{-k} + 1} = \frac{9{,}500}{18e^{-k} + 1}$$

$$18e^{-k} + 1 = \frac{9,500}{1,500}$$

$$18e^{-k} = \frac{95^{\,19}}{\cancel{15}_{\,3}} - 1 \quad \text{OR} \quad 18e^{-k} = \frac{19}{3} - 1$$

$$e^{-k} = \frac{1}{18}\left(\frac{16}{3}\right) \qquad k = -\ln\left(\frac{16}{18.3}\right) = \underline{\underline{1.2164}}$$

Thus $$P(t) = \frac{9,500}{1 + 18e^{-1.2164t}}$$

Finally we need to find t such that

$$4,750 = P(t) = \frac{9,500}{1 + 18e^{-1.2164t}}$$

$$\Longleftrightarrow \quad 1 + 18e^{-1.2164t} = \frac{9,500}{4,750} = 2$$

$$18e^{-1.2164t} = 1$$

$$e^{-1.2164t} = 1/18$$

$$t = -\frac{1}{1.2164}\ln\left(\frac{1}{18}\right) = \frac{\ln(18)}{1.2164}$$

$$\approx \underline{\underline{2.3762}} \text{ } \underline{\text{years}}$$