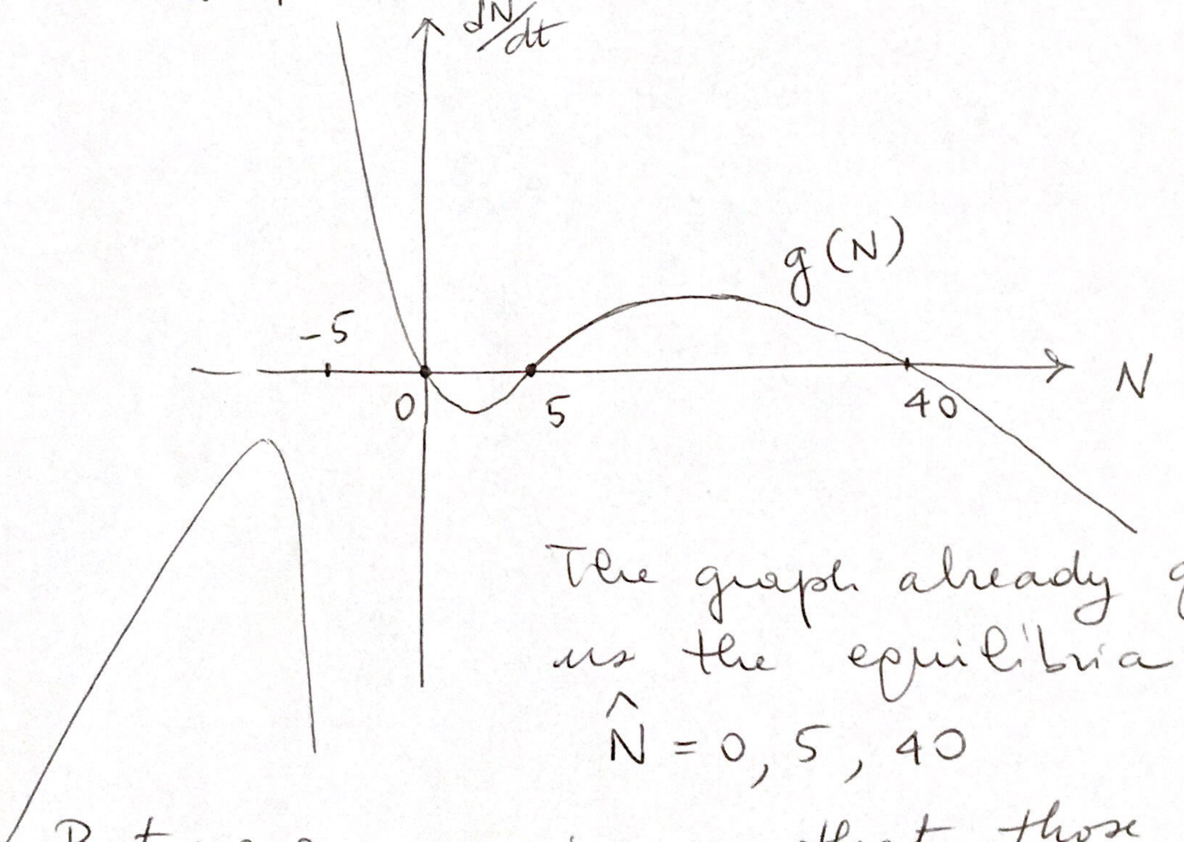
WORKSHEET #14

#1

$$\frac{dN}{dt} = \underbrace{N\left(1 - \frac{N}{50}\right)}_{\text{logistic growth}} - \underbrace{\frac{9N}{5+N}}_{\text{predation}}$$

using a graphing calculator you will see that the graph looks like:



The graph already gives us the equilibria

$$\hat{N} = 0, 5, 40$$

But we can convince us that those values are correct by solving

$$\underbrace{N\left(1 - \frac{N}{50}\right) - \frac{9N}{5+N}}_{g(N)} = 0$$

$$N\left(\frac{50-N}{50}\right) - \frac{9N}{5+N} = 0$$ cross multiply

$$\Leftrightarrow \quad N(5+N)(50-N) - 9N \cdot 50 = 0$$

$$N\left[250 - 5N + 50N - N^2 - 450\right] = 0$$

$$N\left[-N^2 + 45N - 200\right] = 0$$

$$N\left[N^2 - 45N + 200\right] = 0$$

$$N\left[(N-40)(N-5)\right] = 0 \qquad \hat{N} = 0, 5, 40$$

To use the method of eigenvalues, we need to compute $g'(N)$.

$$g(N) = N - \frac{N^2}{50} - \frac{9N}{5+N} \Rightarrow g'(N) = 1 - \frac{N}{25} - \frac{9(5+N) - 9N}{(5+N)^2}$$

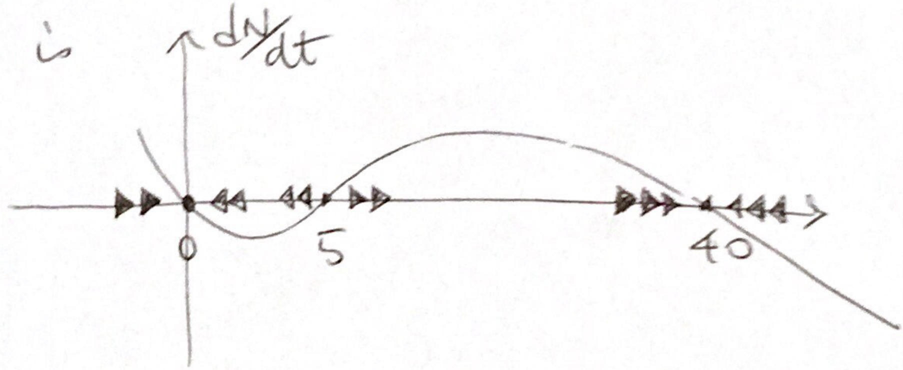
$$g'(N) = \frac{(25-N)(5+N)^2 - 45 \cdot 25}{25(5+N)^2}$$

$$g'(N) = \frac{-(N^3 - 15N^2 - 225N + 500)}{25(5+N)^2}$$

$$\left.\begin{aligned} g'(0) &= \frac{-500}{25 \cdot 25} = -0.8 \\ g'(40) &= -0.62\overline{2} \end{aligned}\right\}$$ hence $\hat{N} = 0, 40$ are locally stable

$g'(5) = 0.35$ hence $\hat{N} = 5$ is unstable

From the graph of $g(N)$ and the sign of $\frac{dN}{dt}$ we conclude that the phase line is



as $N(t)$ is increasing when $\frac{dN}{dt} > 0$

$N(t)$ is decreasing when $\frac{dN}{dt} < 0$

$\boxed{\#2}$

$$\frac{dN}{dt} = \underbrace{2N\left(1 - \frac{N}{1{,}000}\right) - 100}_{g(N)}$$

$$g(N) = 2N - \frac{N^2}{500} - 100 = 0 \iff$$

$$1{,}000N - N^2 - 50{,}000 = 0$$

$$N^2 - 1{,}000N + 50{,}000 = 0$$

use the quadratic formula to obtain

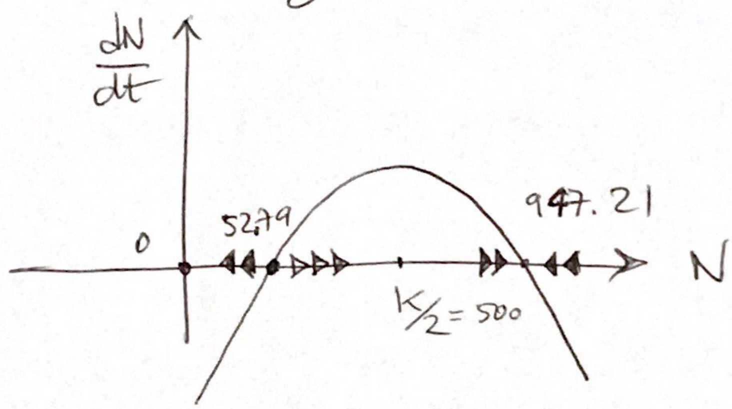
$$\hat{N}_{1,2} = \frac{1000 \pm \sqrt{1000^2 - 4 \cdot 50000}}{2} = \begin{cases} 947.21 \\ 52.79 \end{cases}$$

$$g(N) = 2N - \frac{N^2}{500} - 100$$

so $g'(N) = 2 - \dfrac{N}{250}$

and $g'(52.79) \cong 1.788 > 0$ unstable $\qquad g'(947.21) \cong -1.788$ locally stable

We would have reached the same conclusion using the graphic approach as

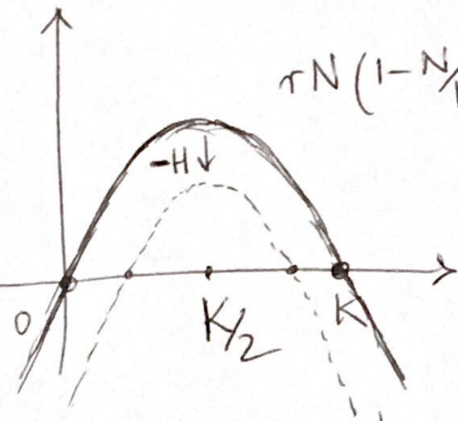


Consider now the general case

$$\frac{dN}{dt} = \underbrace{rN\left(1 - \frac{N}{K}\right) - H}_{g(N)}$$

the graph of $g(N)$ is a parabola $rN\left(1 - \frac{N}{K}\right)$

shifted down by H units



$$\frac{dN}{dt} = rN(1-N/K) - H$$

In order for harvesting to maintain a positive population size we need a locally stable equilibrium, i.e. 2 distinct roots in the 1st quadrant. Either set the discriminant of the equation positive (as in the lecture notes) <u>or</u> ask that the value of the parabola $g(K/2)$ at the vertex is positive:

$$g(K/2) = r(K/2)\left(1 - \frac{K/2}{K}\right) - H > 0$$

$$= r\,K/2\,(1 - 1/2) - H > 0$$

$$= r\,K/4 - H > 0 \quad \text{OR} \quad H < \frac{rK}{4}$$

#3 $\begin{cases} -8x - 7y = 44 \\ 7x + 4y = -30 \end{cases}$

Solve for x in the first equation

$$x = \frac{-7y - 44}{8}$$

and substitute in the second equation

$$7\left(\frac{-7y-44}{8}\right) + 4y = -30$$

$$-49y - 308 + 32y = -240$$

$$-17y = 308 - 240 \qquad y = -\frac{68}{17} = -4$$

Backsubstitute in the original equation

$$x = \frac{-7(-4) - 44}{8} = -2$$

Using elimination:

$$\begin{cases} -8x - 7y = 44 \\ 7x + 4y = -30 \end{cases}$$

Add $\frac{7}{8}$ times the first equation to the second one we obtain

$$\begin{cases} -8x-7y=44 \\ 7x+4y+\boxed{\frac{7}{8}(-8x-7y)}=-30+\boxed{\frac{7}{8}(44)} \end{cases}$$

the second equation becomes

$$4y-\frac{49}{8}y=-30+\frac{77}{2}$$

or
$$-\frac{17}{8}y=\frac{17}{2}$$

or
$$y=-4$$

So
$$\begin{cases} -8x-7y=44 \\ y=-4 \end{cases}$$

Now add 7 times equation 2 to equatio one

$$\begin{cases} -8x-7y\boxed{+7y}=44+\boxed{7(-4)} \\ y=-4 \end{cases}$$

So $\begin{cases} -8x=16 \\ y=-4 \end{cases}$ OR $\begin{cases} x=-2 \\ y=-4 \end{cases}$

$\boxed{\#4}$

$$\begin{cases} -12x + 15y = 6 \\ 20x + ky = 12 \end{cases}$$

equivalent system

$$\begin{cases} x - \frac{15}{12}y = -\frac{6}{12} \\ 20x + ky = 12 \end{cases} \quad \text{OR} \quad \begin{cases} x - \frac{5}{4}y = -\frac{1}{2} \\ 20x + ky = 12 \end{cases}$$

Subtract 20 times the first equation from the second one

$$\begin{cases} x - \frac{5}{4}y = -\frac{1}{2} \\ 0 \cdot x + ky + 20 \cdot \frac{5}{4}y = 12 + 20 \cdot \frac{1}{2} \end{cases}$$

hence the second equation becomes

$$(k+25)y = 22$$

For this equation to have a solution we need $(k+25) \neq 0$.

Hence the system is consistent if $\boxed{k \neq -25}$