WORKSHEET #20

#1

To find the eigenvalues we solve

$$\det \begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix} = 0$$

$$\Longleftrightarrow \quad (3-\lambda)(-1-\lambda) + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

We can write this as

$$\lambda^2 - 2\lambda + 1 = -4$$

or $\quad (\lambda-1)^2 = -4 \quad$ or $\quad \lambda - 1 = \pm\sqrt{-4}$

or $\quad \boxed{\lambda = 1 \pm 2i}$

Let's find the eigenvectors

$\lambda = 1+2i$ $\quad \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (1+2i)\begin{bmatrix} x \\ y \end{bmatrix}$

$$\Longleftrightarrow \quad \begin{cases} 3x - 2y = (1+2i)x \\ 4x - y = (1+2i)y \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} 3x - (1+2i)x - 2y = 0 \\ 4x - y - (1+2i)y = 0 \end{cases}$$

$$\begin{cases} (2-2i)x - 2y = 0 \\ 4x - (2+2i)y = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (1-i)x - y = 0 \\ 2x - (1+i)y = 0 \end{cases}$$

the first equation gives $\boxed{y = (1-i)x}$

the second on is equivalent to

$$(1+i)y = 2x \quad \text{or} \quad y = \frac{2}{1+i}x$$

Which is equivalent to

$$y = \frac{2}{1+i} \cdot \frac{(1-i)}{(1-i)} x = \frac{2(1-i)}{\underbrace{1-i^2}_{2}} x = \underline{\underline{(1-i)x}}$$

which is the same as

So we can choose as eigenvector

$$\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

For $\lambda = 1-2i$ the answer can be chosen to be $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$\boxed{\#2}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = -9\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

are the two matrix equations corresponding to the eigenvectors and eigenvalues.

$$\begin{cases} -2a+4b=-6 \\ -2c+4d=12 \end{cases} \qquad \begin{cases} 3b=0 \\ 3d=-27 \end{cases}$$

Hence $b=0$ and $d=-9$

Hence $-2a+4\cdot 0=-6 \implies a=3$

Hence $-2c+4(-9)=12 \implies c=-24$

$$A = \begin{bmatrix} 3 & 0 \\ -24 & -9 \end{bmatrix}$$

$\boxed{\#3}$ Given $A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$ we have that

the eigenvalues are given solving

$$\det \begin{bmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{bmatrix} = 0$$

$$\Longleftrightarrow \quad (5-\lambda)(-4-\lambda) + 14 = 0$$

$$\lambda^2 - \lambda - 6 = 0 \quad \text{OR} \quad (\lambda-3)(\lambda+2) = 0$$

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = -2$$

<u>Note</u> $\lambda_1 + \lambda_2 = 3-2 = 1 = \text{trace}(A) = 5-4$

$\lambda_1 \cdot \lambda_2 = 3(-2) = -6 = \det(A)$

For the eigenvectors

$\lambda_1 = 3$

$$\begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} 5x + 7y = 3x \\ -2x - 4y = 3y \end{cases} \Longleftrightarrow \begin{cases} 2x + 7y = 0 \\ -2x - 7y = 0 \end{cases}$$

or $\quad y = -\frac{2}{7} x$

As eigenvector we can choose $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$\lambda_2 = -2$

$$\begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -2\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} 5x + 7y = -2x \\ -2x - 4y = -2y \end{cases}$$

$$\Longleftrightarrow \begin{cases} 5x + 5y = 0 \\ -2x - 2y = 0 \end{cases} \quad \text{or} \quad y = -x$$

As eigenvector we can choose $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>Next</u> write $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ as a combination of the eigenvectors. I.e.

$$\begin{bmatrix} -3 \\ -2 \end{bmatrix} = c_1\begin{bmatrix} 7 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or
$$\begin{cases} 7c_1 + c_2 = -3 \\ -2c_1 - c_2 = -2 \end{cases}$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 7 & 1 & -3 \\ -2 & -1 & -2 \end{array}\right]$$

$$\frac{1}{7}R_1 \left[\begin{array}{cc|c} 1 & 1/7 & -3/7 \\ -2 & -1 & -2 \end{array}\right] \rightsquigarrow R_2 + 2R_1 \left[\begin{array}{cc|c} 1 & 1/7 & -3/7 \\ 0 & -5/7 & -\frac{20}{7} \end{array}\right]$$

$$-\frac{7}{5}R_2 \left[\begin{array}{cc|c} 1 & 1/7 & -3/7 \\ 0 & 1 & 4 \end{array}\right] \rightsquigarrow R_1 - 1/7 R_2 \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right]$$

so $c_1 = -1$ and $c_2 = 4$.

<u>Check</u>:

$$\begin{bmatrix} -3 \\ -2 \end{bmatrix} = (-1) \cdot \begin{bmatrix} 7 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence
$$A^{20} \begin{bmatrix} -3 \\ -2 \end{bmatrix} = A^{20} \left[- \begin{bmatrix} 7 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]$$

$$= (-1) A^{20} \begin{bmatrix} 7 \\ -2 \end{bmatrix} + 4 A^{20} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= (-1) \cdot (+3)^{20} \begin{bmatrix} 7 \\ -2 \end{bmatrix} + 4 (-2)^{20} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \cdot 3^{20} + 4 \cdot 2^{20} \\ +2 \cdot 3^{20} - 4 \cdot 2^{20} \end{bmatrix}$$

#4

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -22 \end{bmatrix}$$

$$\underbrace{\qquad}_{A} \cdot \bar{X} = \underline{b}$$

To find the least square solution from the Lecture (28-29) we need to multiply both sides by A^T:

$$\underbrace{\begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 3 & 5 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ -2 \\ -22 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 13 \\ 13 & 27 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -54 \\ -122 \end{bmatrix}$$

To solve it one can multiply both sides by the inverse of the matrix $\begin{bmatrix} 11 & 13 \\ 13 & 27 \end{bmatrix}$. So that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11 & 13 \\ 13 & 27 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -54 \\ -122 \end{bmatrix}$$

Hence

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \underbrace{\frac{1}{11\cdot 27 - 13^2} \cdot \begin{bmatrix} 27 & -13 \\ -13 & 11 \end{bmatrix}}_{\text{inverse}} \cdot \begin{bmatrix} -54 \\ -122 \end{bmatrix}$$

$$= \frac{1}{128} \begin{bmatrix} -27\cdot 54 + 13\cdot 122 \\ 13\cdot 54 - 11\cdot 122 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$x_1^* = 1$; $x_2^* = 5$

$\boxed{\#5}$ From the data in the table we have that $n = c_0 + c_1 p$ leads to the 3 equations

$$\begin{cases} 34 = c_0 + c_1 \cdot 8 \\ 20 = c_0 + c_1 \cdot 11 \\ 13 = c_0 + c_1 \cdot 15 \end{cases}$$

In matrix form: $\begin{bmatrix} 1 & 8 \\ 1 & 11 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 34 \\ 20 \\ 13 \end{bmatrix}$

Multiply by the transpose of $\begin{bmatrix} 1 & 8 \\ 1 & 11 \\ 1 & 15 \end{bmatrix}$ and we obtain

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 8 & 11 & 15 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 1 & 11 \\ 1 & 15 \end{bmatrix}} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 11 & 15 \end{bmatrix} \begin{bmatrix} 34 \\ 20 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 34 \\ 34 & 410 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 67 \\ 687 \end{bmatrix}$$

Multiply both sides of the equation by the inverse of $\begin{bmatrix} 3 & 34 \\ 34 & 410 \end{bmatrix}$ to get

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 3 & 34 \\ 34 & 410 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 67 \\ 687 \end{bmatrix}$$

$$= \frac{1}{3 \cdot 410 - 34^2} \begin{bmatrix} 410 & -34 \\ -34 & 3 \end{bmatrix} \begin{bmatrix} 67 \\ 687 \end{bmatrix}$$

$$= \frac{1}{74} \begin{bmatrix} 410 \cdot 67 - 34 \cdot 687 \\ -34 \cdot 67 + 3 \cdot 687 \end{bmatrix}$$

$$= \begin{bmatrix} 55.57 \\ -2.93 \end{bmatrix}$$

OR $\boxed{n = 55.57 - 2.93 \cdot p}$