WORKSHEET #21

#1 $f(t) = c_0 + c_1 t$

evaluate f at the given data points $(-9,-57)$, $(0,3)$, $(9,51)$. We obtain

$$\begin{cases} c_0 - 9c_1 = -57 \\ c_0 = 3 \\ c_0 + 9c_1 = 51 \end{cases}$$

or in matrix form

$$\begin{bmatrix} 1 & -9 \\ 1 & 0 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -57 \\ 3 \\ 51 \end{bmatrix}$$

You can check that the system has no solution. Multiply by A^T:

$$\begin{bmatrix} 1 & 1 & 1 \\ -9 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & -9 \\ 1 & 0 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -9 & 0 & 9 \end{bmatrix} \begin{bmatrix} -57 \\ 3 \\ 51 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} 3 & 0 \\ 0 & 162 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -3 \\ 972 \end{bmatrix}$$

So $c_0 = -1$ and $c_1 = \frac{972}{162} \approx \underline{6}$

$\boxed{\#2}$ Fit the data to $y = at + b$

We get

$8 = a + b$

$6 = 2a + b$

$3 = 3a + b$

$1 = 4a + b$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \\ 1 \end{bmatrix}$$

Multiply by A^T on both sides

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 33 \\ 18 \end{bmatrix}$$

Multiply by the inverse of $\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$ to get

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{120-100} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} 33 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 4/20 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 33 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} -2.4 \\ 10.5 \end{bmatrix}$$

So $\boxed{y = -2.4t + 10.5}$ is the best fit.

$\boxed{\#3}$ This problem is similar to the computation of the Fibonacci's #

$$\boxed{a_0 = 3, \quad a_1 = 2, \quad a_{k+1} = -2a_{k-1} + 3a_k \text{ for } k \geq 1}$$

We can write this information in matrix form

$$\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a_{k-1} \\ a_k \end{bmatrix}$$

The first equation is a tautology

$$\boxed{a_k = a_k}$$

the second equation is

$$\boxed{a_{k+1} = -2\,a_{k-1} + 3\,a_k}$$

For example

$k=1$ $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$k=2$ $$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}^2 \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$k=3$ $$\begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}^3 \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

So

$$\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}^k \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

To compute $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}^k \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ we need

1. eigenvalues of $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$
2. eigenvectors of $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$
3. write $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a combination of the eigenvectors.

$$\det \begin{bmatrix} 0-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} = 0$$

$$-\lambda(3-\lambda) + 2 = 0 \qquad \lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0 \iff \lambda_1 = 1, \ \lambda_2 = 2$$

The eigenvectors are:

$\lambda_1 = 1$ $\quad \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$

$$\iff \begin{cases} y = x \\ -2x + 3y = y \end{cases} \quad \text{or} \quad \begin{cases} x - y = 0 \\ -2x + 2y = 0 \end{cases}$$

$\iff$ $y = x$ we can choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$ $\quad \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Longleftrightarrow \quad \begin{cases} y = 2x \\ -2x + 3y = 2y \end{cases}$$

Hence there is only one equation

$$y = 2x$$

We can pick as eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Finally write $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a combination of the 2 eigenvectors:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

or $\quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

and $\quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2-1} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

or $\quad \boxed{\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Thus

$$\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}^k \cdot \left(4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

$$= 4 \cdot (1)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2^k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 2^k \\ 4 - 2^k \cdot 2 \end{bmatrix}$$

Thus $\boxed{a_k = 4 - 2^k}$

notice that $a_{k+1} = 4 - 2^{k+1}$

Note $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{4-2^{k+1}}{4-2^k} =$

$$\lim_{k\to\infty} \frac{\overset{\to 0}{4/2^k} - 2}{\underset{\to 0}{4/2^k} - 1} = \boxed{2}$$

or $\lim_{k\to\infty} \frac{a_k}{a_{k+1}} = 1/2$

$\boxed{\#4}$ $h(x,y,z) = \dfrac{xz}{y^4}$

$$h(2,2,3) = \frac{2 \cdot 3}{2^4} = \frac{3}{8}$$

$$h(4,2,2) = \frac{4 \cdot 2}{2^4} = \frac{1}{2}$$

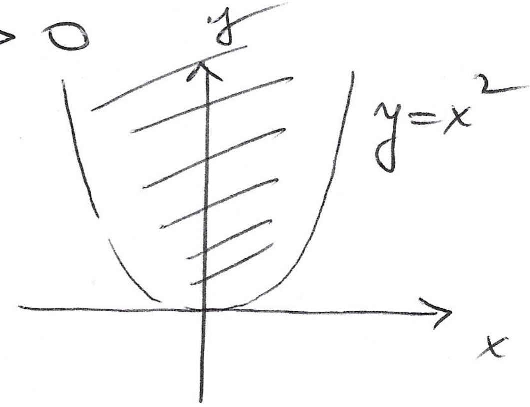
$\boxed{\#5}$ (a) To be defined

$f(x,y) = \ln(y - x^2)$

we need the argument of ln to be positive

domain: $y - x^2 > 0$

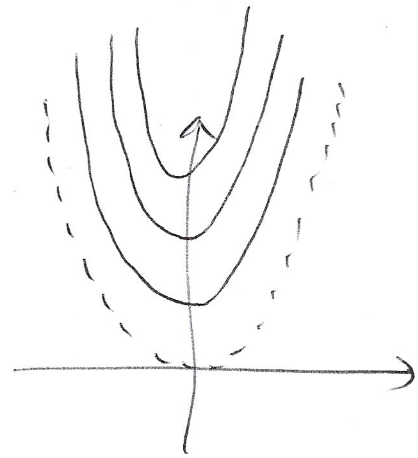
or $y > x^2$



Level curves :

$$f(x,y) = c \iff \ln(y-x^2) = c$$

or $y - x^2 = e^c$



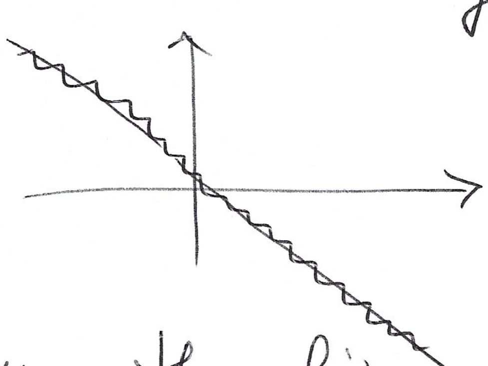
or $\boxed{y = x^2 + \underline{\underline{e^c}}}$

the level curves are shifts of the parabola $y = x^2$

c could be any real number.

(b) $f(x,y) = \dfrac{x-y}{x+y}$ the domain is all (x,y) such that $x+y \neq 0$

or $y \neq -x$



all plane <u>minus</u> the line $y = -x$.

For the level curves, we need to solve

$$\frac{x-y}{x+y} = c$$

$$x - y = c(x+y)$$

$$\Longleftrightarrow$$

$$x - y = cx + cy$$

Solve for y:

$$cy + y = x - cx$$

$$y(1+c) = x(1-c)$$

or $$y = \frac{1-c}{1+c} \cdot x$$

the level curves are lines through the origin with slope $\frac{1-c}{1+c}$

c cannot be -1.