WORKSHEET #27

$\boxed{\#1}$ Use a direct substitutions method

The pair (f) and (g) satisfy the system of DE

(f) $y_1 = e^x$ and $y_2 = e^x$

1st eq.

$$(e^x)' \stackrel{?}{=} 2.5(e^x) - 1.5(e^x)$$

$$e^x \stackrel{?}{=} e^x \quad \underline{\underline{\text{Yes}}}$$

2nd eq

$$\underbrace{(e^x)'}_{e^x} \stackrel{?}{=} -1.5(e^x) + 2.5(e^x)$$

$$e^x \stackrel{?}{=} 1 \cdot e^x \quad \underline{\underline{\text{Yes}}}$$

(g) $y_1 = e^{4x}$, $y_2 = -e^{4x}$

1st eq

$$\underbrace{(e^{4x})'}_{4e^{4x}} \stackrel{?}{=} \underbrace{2.5(e^{4x}) - 1.5(-e^{4x})}_{4e^{4x}}$$

$$4e^{4x} \stackrel{?}{=} 4e^{4x} \quad \text{Yes}$$

$$\underbrace{(-e^{4x})'}_{-4e^{4x}} \stackrel{?}{=} \underbrace{-1.5(+e^{4x}) + 2.5(-e^{4x})}_{-4e^{4x}}$$

$$-4e^{4x} \stackrel{?}{=} -4e^{4x} \quad \text{Yes}$$

$\boxed{\#2}$

$$\begin{cases} x_1' = 4x_1 - 7x_2 \\ x_2' = 2x_1 - 5x_2 \end{cases}$$

OR $$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{trace}(A) = 4 - 5 = \underline{\underline{-1 = \lambda_1 + \lambda_2}}$$

$$\det(A) = -20 + 14 = \underline{\underline{-6 = \lambda_1 \lambda_2}}$$

Hence $\boxed{\lambda_1 = -3}$ and $\boxed{\lambda_2 = 2}$

This means that the origin is a <u>saddle</u> (unstable) <u>equilibrium</u>

The eigenvectors are:

$\lambda_1 = -3$ $$\begin{bmatrix} 4 & -7 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{cases} 4v_1 - 7v_2 = -3v_1 \\ 2v_1 - 5v_2 = -3v_2 \end{cases} \iff \underline{\underline{v_1 = v_2}}$$

Thus we can choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as eigenvectors

$\lambda_2 = 2$ $$\begin{bmatrix} 4 & -7 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{cases} 4v_1 - 7v_2 = 2v_1 \\ 2v_1 - 5v_2 = 2v_2 \end{cases} \Longleftrightarrow \quad 2v_1 = 7v_2$$

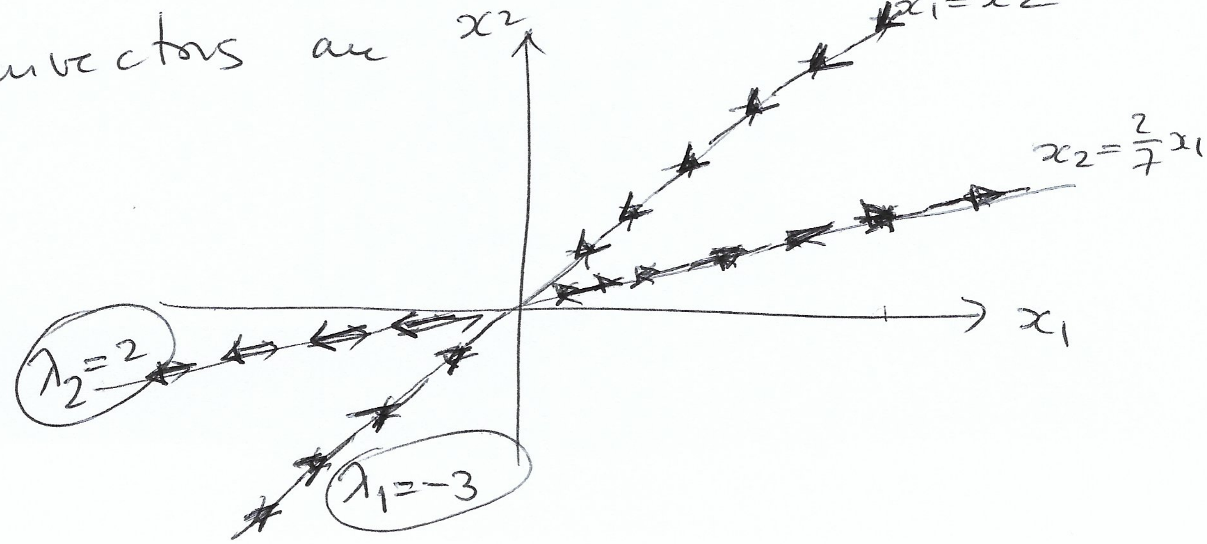
or $v_2 = \frac{2}{7} v_1$

Thus we can pick $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ as eigenvector

the general solution looks like

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 7 \\ 2 \end{bmatrix} e^{2t}$$

The direction along the lines of eigenvectors are



$\boxed{\#3}$ $\frac{d}{dt}\underline{x} = \begin{bmatrix} 22 & -54 \\ 9 & -23 \end{bmatrix}\underline{x}$

with $\underline{x}(0) = \begin{bmatrix} 16 \\ 7 \end{bmatrix}$

the eigenvalues of the matrix A satisfy

$$\lambda_1 + \lambda_2 = 22 - 23 = -1$$

$$\lambda_1 \cdot \lambda_2 = 22(-23) - (-54)(9)$$

$$= -20$$

So $\underline{\lambda_1 = -5}$ and $\underline{\lambda_2 = 4}$

The eigenvectors are:

$$\begin{bmatrix} 22 & -54 \\ 9 & -23 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

gives the relation $v_1 = 2v_2$

so we can pick $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for example

$$\begin{bmatrix} 22 & -54 \\ 9 & -23 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

gives the relation $v_1 = 3v_2$ so we can pick $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Thus the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{4t}$$

When $t = 0$

$$\begin{bmatrix} 16 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

as $\underline{\underline{e^0 = 1}}$. Or $\begin{cases} 2c_1 + 3c_2 = 16 \\ c_1 + c_2 = 7 \end{cases}$

Solve the system to get $c_1 = 5$
$c_2 = 2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t} + 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{4t}$$

or $\begin{cases} x_1(t) = 10e^{-5t} + 6e^{4t} \\ x_2(t) = 5e^{-5t} + 2e^{4t} \end{cases}$

$\boxed{\#4}$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

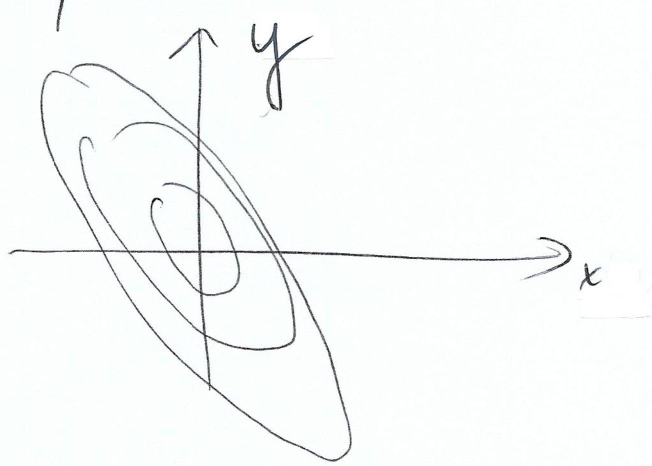
$$\det \begin{bmatrix} 3-\lambda & 2 \\ -5 & -3-\lambda \end{bmatrix} = 0$$

OR $\quad (3-\lambda)(-3-\lambda) + 10 = 0$

$$\lambda^2 - 9 + 10 = 0$$

OR $\quad \lambda^2 = -1 \quad$ hence $\quad \lambda = \pm\sqrt{-1} = \pm i$

This means the $(0,0)$ is a neutral spiral or center. From the direction field the trajectories are closed ellipses



To find the eigenvectors (complex conjugate)

$$\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = i \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{cases} 3x + 2y = ix \\ -5x - 3y = iy \end{cases} \quad \text{OR} \quad \begin{cases} (3-i)\,x + 2y = 0 \\ -5x - (3+i)\,y = 0 \end{cases}$$

We get the same equation

$$y = \frac{i-3}{2}\, x$$

(second equation gives $y = -\frac{5}{3+i}\, x$

but if you multiply top and bottom by $3-i$ you get

$$y = -\frac{5(3-i)}{(3+i)(3-i)}\, x = \frac{-5(3-i)}{10}\, x = \frac{i-3}{2}\, x \Big)$$

Choose $\begin{bmatrix} 2 \\ i-3 \end{bmatrix}$ as eigenvector.

The complex solution is

$$\begin{bmatrix} 2 \\ i-3 \end{bmatrix} e^{it}$$

If we use Euler's Formula

$$\left(\begin{bmatrix}2\\-3\end{bmatrix}+i\begin{bmatrix}0\\1\end{bmatrix}\right)\cdot(\cos t+i\sin t)$$

$$=\left(\begin{bmatrix}2\\-3\end{bmatrix}\cos t-\begin{bmatrix}0\\1\end{bmatrix}\sin t\right)+i\left(\begin{bmatrix}0\\1\end{bmatrix}\cos t+\begin{bmatrix}2\\-3\end{bmatrix}\sin t\right)$$

$$=\begin{bmatrix}2\cos t\\-3\cos t-\sin t\end{bmatrix}+i\begin{bmatrix}2\sin t\\\cos t-3\sin t\end{bmatrix}$$

Hence the general solution is

$$\begin{bmatrix}x\\y\end{bmatrix}=c_1\begin{bmatrix}2\cos t\\-3\cos t-\sin t\end{bmatrix}+c_2\begin{bmatrix}2\sin t\\\cos t-3\sin t\end{bmatrix}$$