WORKSHEET #29

#1 The system of DE described in Figure 11.40 is

arrows going out

$$\frac{dx_1}{dt} = I - ax_1 - cx_1 + bx_2$$

arrows going in

$$\frac{dx_2}{dt} = ax_1 - bx_2 - dx_2$$

arrow going in

arrows going out

with our information

$$\frac{dx_1}{dt} = -2.5\, x_1 + 0.7\, x_2$$

$$\frac{dx_2}{dt} = 2.5\, x_1 - 0.8\, x_2$$

In Matrix form: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2.5 & 0.7 \\ 2.5 & -0.8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

The eigenvalues are

$$\det \begin{bmatrix} -2.5-\lambda & 0.7 \\ 2.5 & -0.8-\lambda \end{bmatrix} = 0$$

$$(-2.5-\lambda)(-0.8-\lambda) - 2.5 \cdot 0.7 = 0$$

$$\lambda^2 + 3.3\lambda + 2.5 = 0$$

i.e $\quad \mathrm{trace}(A) = -3.3$

$\quad \det(A) = 2.5$

This means that the eigenvalues are both positive or both negative (real part)

But the trace is negative so they must be both negative

Explicitly $\quad \lambda_{1,2} = \dfrac{-3.3 \pm \sqrt{3.3^2 - 4 \cdot 2.5}}{2}$

$$= \begin{cases} -1.1783 \\ -2.121699 \end{cases}$$

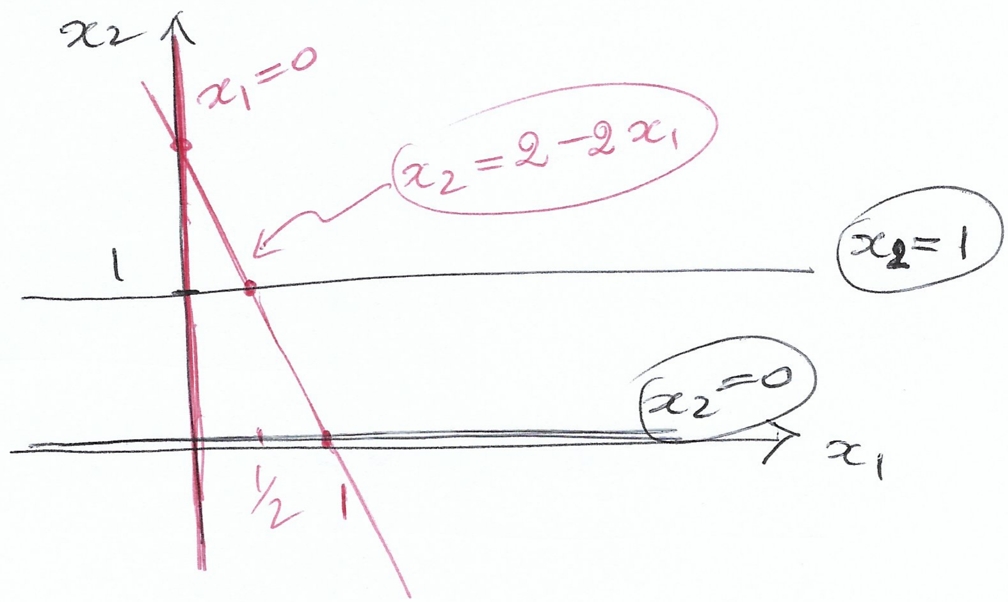
Hence $(0,0)$ is a stable node

$\boxed{\#2}$

$$\begin{cases} \dfrac{dx_1}{dt} = x_1\left[4 - 4x_1 - 2x_2\right] \\ \\ \dfrac{dx_2}{dt} = x_2(1 - x_2) \end{cases}$$

I factored out in each term either x_1 or x_2.

The nullclines are:



The equilibria are at the points where both $\frac{dx_1}{dt} = 0$ and $\frac{dx_2}{dt} = 0$ simultaneously. Hence we have four equilibria

$(0,0)$; $(1,0)$; $(0,1)$ and $(\frac{1}{2}, 1)$

The Jacoby matrix

$$Df(x_1,x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

is

$$= \begin{bmatrix} 4 - 8x_1 - 2x_2 & -2x_1 \\ 0 & 1 - 2x_2 \end{bmatrix}$$

Let's evaluate it at the three equilibria

$$Df(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

eigenvalues are
4, 1

hence $(0,0)$ is
an unstable node

$$Df(1,0) = \begin{bmatrix} -4 & -2 \\ 0 & 1 \end{bmatrix}$$

the eigenvalues are
-4, 1

hence $(1,0)$ is
a saddle point

$$Df(1/2, 1) = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}$$

the eigenvalues are $-2, -1$

hence $(1/2, 1)$ is a <u>stable node</u>

$$Df(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

the eigenvalues are $2, -1$

hence $(0,1)$ is a <u>saddle point</u>

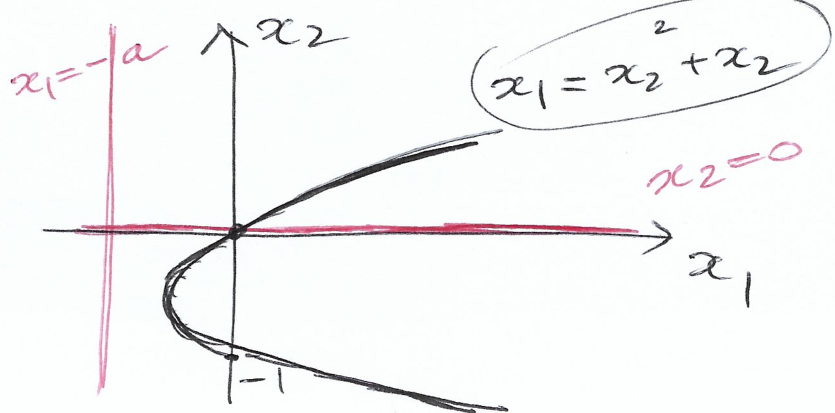
$\boxed{\#3}$

Notice that the null clines of

$$\frac{dx_1}{dt} = x_2(x_1 + a)$$

$$\frac{dx_2}{dt} = x_2^2 + x_2 - x_1$$

are



Notice that the vertex of the parabola is at the point $x_2 = -\frac{1}{2}$ (halfway between $x_2 = 0$ and $x_2 = -1$) and its x_1-coordinate is $x_1 = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) = -\frac{1}{4}$

Thus if we want <u>only one</u> equilibrium we need $-a < -\frac{1}{4}$ <u>or</u> $a > \frac{1}{4}$

For this choice of a, the only equilibrium is $(0,0)$. We need the Jacobi matrix at $(0,0)$.

$$Df(x_1,x_2) = \begin{bmatrix} x_2 & a \\ -1 & 2x_2+1 \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 0 & a \\ -1 & 1 \end{bmatrix}$$

The eigenvalues are:

$$\det \begin{bmatrix} -\lambda & a \\ -1 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda(1-\lambda) + a = 0$$

$$\lambda^2 - \lambda + a = 0 \qquad \left(a > 1/4\right)$$

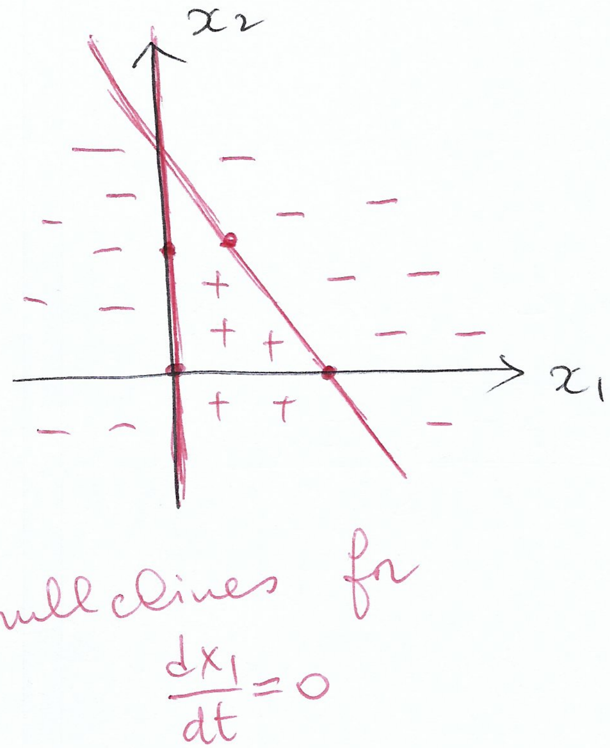
$$\lambda_{1,2} = \frac{1 \pm \sqrt{1-4a}}{2}$$

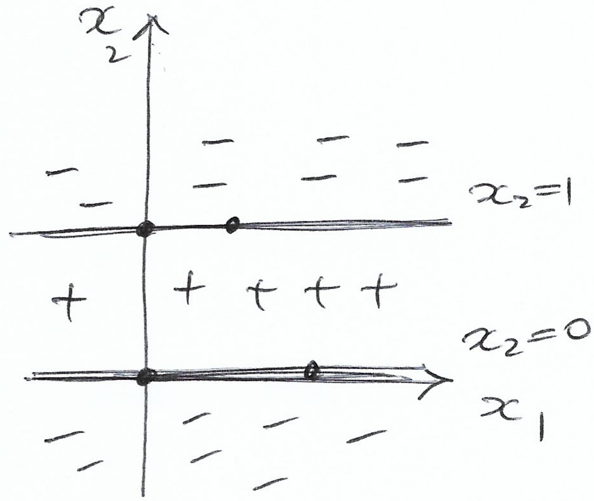
For $a > 1/4$ the discriminant is negative so we have two complex conjugate eigenvalues with positive real part

Hence $(0,0)$ is an <u>unstable spiral</u>

$\boxed{\#4}$ Using the graphical approach.

The nullclines are the same as in $\boxed{\#2}$. Let's look at the sign on either side of the null clines by picking suitable points:





The sign of the Jacobi matrix at:

* $(0,0)$ are: $\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$ hence $(0,0)$ is an unstable node

* $(1,0)$ are: $\begin{bmatrix} - & - \\ 0 & + \end{bmatrix}$ hence $(1,0)$ is a saddle point

* $(0,1)$ are : $\begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}$ hence $(0,1)$ is a <u>saddle point</u>

* $(\frac{1}{2},1)$ are : $\begin{bmatrix} - & - \\ 0 & - \end{bmatrix}$ hence $(\frac{1}{2},1)$ is a <u>stable node</u>