WORKSHEET #6

#1

$$\int \frac{-e^{2x}}{e^{2x}+3e^x+2}\,dx = \int \frac{-e^x \cdot e^x\,dx}{(e^x)^2+3e^x+2}$$

so if we set $u=e^x$; $\frac{du}{dx}=e^x$ or

$du=e^x\,dx$. Thus the integral becomes

$$= \int \frac{-u}{u^2+3u+2}\,du = \int \frac{-u}{(u+2)(u+1)}\,du$$

Use the partial fraction decomposition

$$\frac{-u}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1} = \frac{A(u+1)+B(u+2)}{(u+2)(u+1)}$$

So: $\boxed{-u = A(u+1) + B(u+2)}$

substitute $\boxed{u=-1}$ to obtain

$1 = A\cdot 0 + B(-1+2)$ so $B=1$

substitute $\boxed{u=-2}$ to obtain

$2 = A(-2+1) + B\cdot 0$ $\quad A=-2$

That is $\frac{-u}{(u+2)(u+1)} = \frac{-2}{u+2} + \frac{1}{u+1}$

$$\int \frac{-u}{(u+2)(u+1)}\,du = \int \left(\frac{1}{u+1} - \frac{2}{u+2}\right) du =$$

$$= \ln|u+1| - 2\ln|u+2| + C$$

$$= \ln|e^x+1| - 2\ln|e^x+2| + C$$

they are all positive functions so you can drop $|\cdot|$

$$= \boxed{\ln\left(\frac{e^x+1}{(e^x+2)^2}\right) + C}$$ using properties of $\ln$

$\boxed{\#2}$ $\displaystyle\int \frac{x+2}{x^3-x}\,dx$ this is a proper fraction

The denominator factors as: $x^3 - x = x(x^2-1)$ $= x(x+1)(x-1)$. Thus we seek to find A, B, C such that

$$\frac{x+2}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$= \frac{A(x+1)(x-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)}$$

So $x+2 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$

set $\boxed{x=0}$; we cover B and C.

$$0+2 = A(1)(-1) + B\cdot\cancel{0} + C\cdot\cancel{0}$$

So $\boxed{A=-2}$

set $\boxed{x=1}$; we cover A and B

$$1+2 = A\cdot 0 + B\cdot 0 + C(2)$$

So $\boxed{C = 3/2}$

set $\boxed{x=-1}$; we cover A and C

$$-1+2 = A\cdot\cancel{0} + B(-1)(-2) + C\cdot\cancel{0}$$

so $B = 1/2$

Thus

$$\frac{x+2}{x^3-x} = \frac{-2}{x} + \frac{1/2}{x+1} + \frac{3/2}{x-1}$$

$$\int \frac{x+2}{x^3-x}\,dx = \boxed{-2\ln|x| + \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C}$$

$\boxed{\#3}$ $\displaystyle\int_3^8 \frac{6x}{x^2+4x+4}\,dx$

this is a proper fraction. Notice that the denominator is $\underline{(x+2)^2}$. Thus it has a repeated factor. We seek to decompose

as $\displaystyle\frac{6x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$

<u>So</u>: $6x = A(x+2) + B$

Using cover up we can only find B.

Set $\underline{x=-2}$ $\quad 6(-2) = A\cdot\cancel{(0)}^{\,0} + B$

$\boxed{B=-12}$. To find A substitute any other value for x. Say $x=0$.

$6\cdot 0 = A(2) + B$. Thus $A = -B/2$

$\boxed{A=6}$

$$\frac{6x}{(x+2)^2} = \frac{6}{x+2} - \frac{12}{(x+2)^2}$$

$$\int \frac{6x}{(x+2)^2}\,dx = 6\int \frac{1}{x+2} - 12\int \frac{1}{(x+2)^2}\,dx$$

$$= 6\ln|x+2| - 12\int \frac{1}{u^2}\,du$$

set $u = x+2$

$du = dx$

$$= 6\ln|x+2| - 12\left(-u^{-1}\right) + C$$

$$= \boxed{6\ln|x+2| + \frac{12}{x+2} + C}$$

$$\int_3^8 \frac{6x}{(x+2)^2}\,dx = 6\ln|x+2| + \frac{12}{x+2}\Bigg|_3^8$$

$$= 6\ln(10) + \frac{12}{10} - 6\ln(5) - \frac{12}{5}$$

$$= 6\ln\left(\frac{10}{5}\right) + 12\left(\frac{1}{10} - \frac{1}{5}\right) = \boxed{6\ln 2 - \frac{6}{5}}$$

$$\approx \underline{2.9589}$$

$\boxed{\#4}$ $\displaystyle\int \frac{9x^2}{(x+1)^3}\,dx$

Same procedure as in problem #3.

$$\frac{9x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

so: $9x^2 = A(x+1)^2 + B(x+1) + C$

Set $\boxed{x=-1}$ so we can cover A and B

$$9(-1)^2 = A\cdot 0 + B\cdot 0 + C$$

$\therefore$ $\boxed{C=9}$

set $\boxed{x=0}$ to find one relation between A and B

$$0 = A + B + \overset{9}{\cancel{C}}$$

set $\boxed{x=1}$ to find another relation between A and B

$$9(1)^2 = A\cdot 4 + B\cdot 2 + C$$

Thus $\begin{cases} A + B + 9 = 0 \\ 4A + 2B + \cancel{9} = \cancel{9} \end{cases}$

$\begin{cases} A + B + 9 = 0 \\ B = -2A \end{cases}$

substituting $A - 2A + 9 = 0$ $\quad \boxed{A = 9}$

$\boxed{B = -18}$

So :

$$\int \frac{9x^2}{(x+1)^3}\,dx = \int \frac{9}{x+1}\,dx - \int \frac{18}{(x+1)^2}\,dx + \int \frac{9}{(x+1)^3}\,dx$$

use substitution

$u = x + 1$

$du = dx$

$$= 9 \ln|x+1| + \frac{18}{x+1} + 9\left(-\frac{1}{2}\frac{1}{(x+1)^2}\right) + C$$

$$\boxed{= 9 \ln|x+1| + \frac{18}{x+1} - \frac{9/2}{(x+1)^2} + C}$$

$$= \frac{9}{2} \cdot \frac{4(x+1) - 1}{(x+1)^2} + 9 \ln|x+1| + C =$$

$$= \frac{9}{2} \frac{4x+3}{(x+1)^2} + 9 \ln|x+1| + C$$

$\boxed{\#5}$

$$\frac{x^2+3x-10}{(x^2+4x+6)^2(x^2-1)(x+1)}$$

notice that x^2+4x+6 is irreducible.

It has complex roots:

$$x^2+4x+6=0 \iff (x^2+4x+4)+2=0$$

$$(x+2)^2+2=0 \iff (x+2)^2=-2 \quad \text{so}$$

$$x_{1,2}=-2\pm\sqrt{-2}$$

The factorization of the denominator is

$$\underline{(x^2+4x+6)^2\cdot(x-1)(x+1)^2} \quad \text{as } x^2-1=(x-1)(x+1)$$

So

$$\frac{x^2+3x-10}{(x^2+4x+6)^2(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$+\frac{Dx+E}{x^2+4x+6}+\frac{Fx+G}{(x^2+4x+6)^2}$$

for suitable $A, B, C, \ldots G$.