WORKSHEET #8

#1 $\displaystyle\int_0^1 -\frac{3}{x^7}\,dx$ is an improper integral as the function goes to $-\infty$ at $x=0$

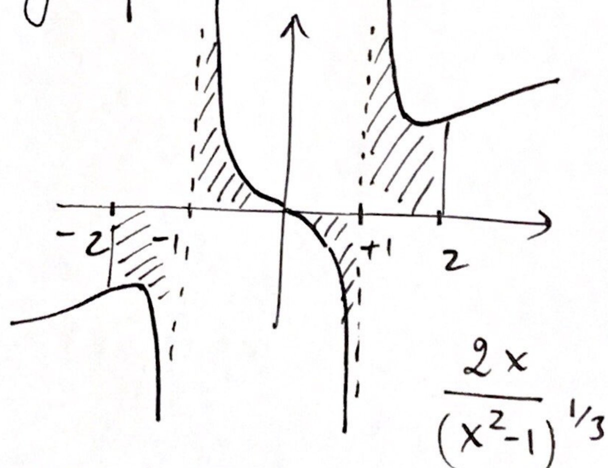
$$= \lim_{a\to 0^+} \int_a^1 -\frac{3}{x^7}\,dx = \lim_{a\to 0^+} \left(-3\frac{1}{(-6)} x^{-7+1} \Big|_a^1 \right)$$

$$= \lim_{a\to 0^+} \left(\frac{1}{2}\cdot\frac{1}{x^6} \Big|_a^1 \right) = \lim_{a\to 0^+} \left(\frac{1}{2} - \frac{1}{2a^6} \right)$$

$= -\infty$ hence the integral diverges.

#2 $\displaystyle\int_{-2}^{2} \frac{2x}{(x^2-1)^{1/3}}\,dx$ the function has vertical asymptotes at $x = \pm 1$

In fact the graph is:



We need to break the integral into 4 parts

$$\boxed{\int_{-2}^{-1} \frac{2x}{(x^2-1)^{1/3}}\,dx + \int_{-1}^{0} \frac{2x}{(x^2-1)^{1/3}}\,dx + \int_{0}^{1} \frac{2x}{(x^2-1)^{1/3}}\,dx + \int_{1}^{2} \frac{2x}{(x^2-1)^{1/3}}\,dx}$$

For the integral to converge each of those integrals need to be finite.

Also, notice that the antiderivative of the function is

$$\int \frac{2x}{(x^2-1)^{1/3}}\,dx \underset{\substack{u=x^2-1 \\ du=2x\,dx}}{=\!=} \int \frac{1}{u^{1/3}}\,du = \int u^{-1/3}\,du$$

$$= \frac{3}{2}u^{-\frac{1}{3}+1} + C = \frac{3}{2}u^{2/3} + C = \frac{3}{2}\sqrt[3]{(x^2-1)^2} + C$$

Let's analyze each integral

$$* \int_{-2}^{-1} \frac{2x}{(x^2-1)^{1/3}}\,dx = \lim_{b\to -1^-} \int_{-2}^{b} \frac{2x}{(x^2-1)^{1/3}}\,dx =$$

$$= \lim_{b\to -1^-}\left(\left. \frac{3}{2}\sqrt[3]{(x^2-1)^2}\,\right|_{-2}^{b} \right) = \lim_{b\to -1^-}\left(\underbrace{\frac{3}{2}\sqrt[3]{(b^2-1)^2}}_{\to 0} - \right.$$

$$\left. - \frac{3}{2}\sqrt[3]{(4-1)^2}\right) = 0 - \frac{3}{2}\sqrt[3]{9} \approx \underline{-3.1201}$$

$$* \int_{-1}^{0} \frac{2x}{(x^2-1)^{1/3}}\,dx = \lim_{a\to -1^+} \int_{a}^{0} \frac{2x}{(x^2-1)^{1/3}}\,dx = \lim_{a\to -1^+}\left(\left. \frac{3}{2}\sqrt[3]{(x^2-1)^2}\,\right|_{a}^{0} \right)$$

$$= \lim_{a\to -1^+}\left(\frac{3}{2} - \underbrace{\frac{3}{2}\sqrt[3]{(a^2-1)^2}}_{\to 0} \right) = \frac{3}{2}$$

$* \displaystyle\int_0^1 \frac{2x}{(x^2-1)^{1/3}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{2x}{(x^2-1)^{1/3}}\,dx =$

$$= \lim_{b\to 1^-}\left(\frac{3}{2}\sqrt[3]{(x^2-1)^2}\,\Big|_0^b \right) = \lim_{b\to 1^-}\Big[\frac{3}{2}\underbrace{\sqrt[3]{(b^2-1)^2}}_{\to 0} - 3/2\Big]$$

$= -3/2$

$* \displaystyle\int_1^2 \frac{2x}{(x^2-1)^{1/3}}\,dx = \lim_{a\to 1^+} \int_a^2 \frac{2x}{(x^2-1)^{1/3}}\,dx =$

$$= \lim_{a\to 1^+}\left(3/2\sqrt[3]{(x^2-1)^2}\,\Big|_a^2 \right) = \lim_{a\to 1^+}\Big(3/2\sqrt[3]{9} - \underbrace{\frac{3}{2}\sqrt[3]{(a^2-1)^2}}_{\to 0}\Big)$$

$= \frac{3}{2}\sqrt[3]{9} \approx 3.1201$

Hence $\displaystyle\int_{-2}^{2} \frac{2x}{(x^2-1)^{1/3}}\,dx$ converges because each of the four integrals exist and its value is the sum of those 4 values; hence $\boxed{=0}$

$\boxed{\#3}$ For $x \geq 1$ we have that

$$\underbrace{x\cdot x \geq x\cdot 1}_{\text{multiply by } x\geq 1} \quad \text{or} \quad -x^2 \leq -x$$

(multiply by -1)

But exp is an increasing function so

$$0 \leq e^{-x^2} \leq e^{-x} \quad \text{for } x \geq 1$$

Thus $0 \leq \displaystyle\int_1^{\infty} e^{-x^2}\,dx \leq \int_1^{\infty} e^{-x}\,dx$

and $\displaystyle\int_1^{\infty} e^{-x}\,dx = \lim_{b\to+\infty} \int_1^{b} e^{-x}\,dx =$

$$= \lim_{b\to+\infty} \left(-e^{-x} \Big|_1^b \right) = \lim_{b\to\infty} \left(-e^{-b} - (-e^{-1}) \right)$$

$$= \lim_{b\to\infty} \left(\frac{1}{e} - \underbrace{e^{-b}}_{\to 0} \right) = \frac{1}{e}$$

So $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ converges and its value is $\leq \frac{1}{e}$.

$\boxed{\#4}$ For $x > 0$ $\quad x^4 \leq \underbrace{1+x^4}_{\text{we add } 1}$

and $\sqrt{x^4} \leq \sqrt{1+x^4}$ because $\sqrt{\cdots}$ is an increasing function. So $x^2 \leq \sqrt{1+x^4}$

and $0 \leq \dfrac{1}{\sqrt{1+x^4}} \leq \dfrac{1}{x^2}$

Thus

$$0 \leq \int_1^\infty \frac{1}{\sqrt{1+x^4}}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx$$

But $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx =$

$$= \lim_{b\to\infty}\int_1^b x^{-2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left[-\frac{1}{b} - (-1)\right] = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = \boxed{1}$$

Hence $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^4}}\,dx$ converges and its value is less than 1.