# MA 138 Worksheet #12

Handout & Section 8.2

2/15/24

## Direction fields

A **direction field** (or **slope field**) is a graphical representation of the solutions of a first-order differential equation of the form

$$\frac{dy}{dx} = f(x,y).$$

We (or, better, a computer) can construct a direction field (or slope field) by evaluating the function $f(x,y)$ at each point of a rectangular grid consisting of at least a few hundred points. Then, at each point of the grid, a short line segment is drawn whose slope is the value of $f$ at that point.

Thus each line segment is tangent to the graph of the solution passing through that point.

A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a given differential equation.

## SageMathCell

The SageMathCell project is an easy-to-use web interface of the free open-source mathematics software system Sage. Go to https://sagecell.sagemath.org/ and just type some Sage code in the provided box and then press Evaluate. For example the commands

```
x,y=var('x,y')
plot_slope_field(x^2*y^2,(x,-5,5),(y,-10,10), headaxislength=3, headlength=3)
```

produce the direction field of the differential equation $\frac{dy}{dx} = x^2y^2$.

**1** Draw the direction fields of the following differential equations:

$(a)$ $y' = y + 2$

$(b)$ $y' = -2 + x - y$

$(c)$ $y' = 2\sin(x) + 1 + y$

**Equilibria of an autonomous differential equation**

If $\widehat{y}$ (read "$y$ hat") satisfies

$$g(\widehat{y}) = 0$$

then $\widehat{y}$ is an equilibrium of the autonomous differential equation

$$\frac{dy}{dx} = g(y).$$

The basic property of equilibria is that if, initially (say, at $x = 0$), $y(0) = \widehat{y}$ and $\widehat{y}$ is an equilibrium, then $y(x) = \widehat{y}$ for all $x \geq 0$.

**Analytic Approach to Stability: "Stability Criterion"**

Consider the differential equation $\frac{dy}{dx} = g(y)$ where $g(y)$ is a differentiable function.
Assume that $\widehat{y}$ is an equilibrium; that is, $g(\widehat{y}) = 0$.
Then

- $\widehat{y}$ is **locally stable** if $g'(\widehat{y}) < 0$;
- $\widehat{y}$ is **unstable** if $g'(\widehat{y}) > 0$.

**Note:** $g'(\widehat{y})$ is called an **eigenvalue**; it is the slope of the tg. line of $g(y)$ at $\widehat{y}$.

**2** Suppose that $\frac{dy}{dx} = (4 - y)(5 - y)$.

$(a)$ Find the equilibria of this differential equation.

$(b)$ Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

$(c)$ Plot the direction field associated with this differential equation.

$(d)$ Solve the differential equation and compute the limit as $x$ approaches infinity of the solution $y = y(x)$.