MA 138 Worksheet #13

Section 8.2

2/20/24

Equilibria of an autonomous differential equation

If $\widehat{y}$ (read "y hat") satisfies $g(\widehat{y}) = 0$ then $\widehat{y}$ is an **equilibrium of the autonomous differential equation** $\dfrac{dy}{dx} = g(y)$.

The **basic property** of equilibria is that if, initially (say, at $x = 0$), $y(0) = \widehat{y}$ and $\widehat{y}$ is an equilibrium, then $y(x) = \widehat{y}$ for all $x \geq 0$.

Analytic Approach to Stability: "Stability Criterion"

Consider the differential equation $\dfrac{dy}{dx} = g(y)$ where $g(y)$ is a differentiable function.
Assume that $\widehat{y}$ is an equilibrium; that is, $g(\widehat{y}) = 0$.
Then

- $\widehat{y}$ is **locally stable** if $g'(\widehat{y}) < 0$;
- $\widehat{y}$ is **unstable** if $g'(\widehat{y}) > 0$.

Note: $g'(\widehat{y})$ is called an **eigenvalue**; it is the slope of the tg. line of $g(y)$ at $\widehat{y}$.

1 Suppose that $\dfrac{dy}{dx} = y(y-1)(y-2)$.

(a) Find the equilibria of this differential equation.

(b) Compute the eigenvalues associated with each equilibrium and discuss the stability of the equilibria.

2 Two different strains of bacteria sometimes feed on chemicals excreted by one another: strain 1 feeds on chemicals produced by strain 2, and vice versa. This phenomenon is referred to as *cross-feeding*. For a relatively simple model of cross-feeding, it can be shown that the frequency $P(t)$ of the strain 1 bacteria is governed by the differential equation

$$\frac{dP}{dt} = P(1-P)(\alpha(1-P) - \beta P),$$

where α and β are positive constants.

(a) Assume $\alpha = 2$ and $\beta = 3$. Write down your new differential equation.

(b) Find the stable equilibrium point(s).

(c) Create a slope field for the differential equation.

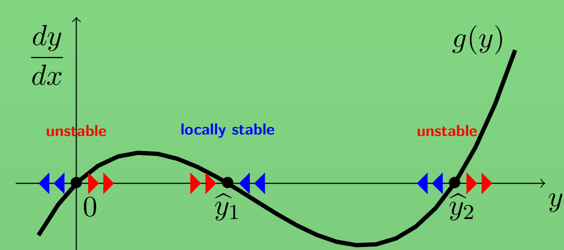
Graphical Approach to Stability

Consider the autonomous differential equation $\dfrac{dy}{dx} = g(y).$

To find the equilibria of our DE, we set $g(y) = 0$. **Graphically**, this means that if we graph $g(y)$ (i.e., the derivative of y with respect to x) as a function of y, then the equilibria are the points of intersection of $g(y)$ with the horizontal axis, which is the y-axis in this case, since y is the independent variable.

We can then use the graph of $g(y)$ to say the following about the fate of a solution on the basis of its current value:

- if the current value y is such that $g(y) > 0$ (i.e., $dy/dx > 0$), then y will increase as a function of x;
- if the current value y is such that $g(y) < 0$ (i.e., $dy/dx < 0$), then y will decrease as a function of x;
- the points y where $g(y) = 0$ are the points where y will not change as a function of x [since $g(y) = dy/dx = 0$]. These are the equilibria.



The arrows close to the equilibria indicate the type of stability. For our choice of $g(y)$, the equilibria are at $\widehat{y} = 0$, $\widehat{y}_1$, and $\widehat{y}_2$.

3 Suppose that $\dfrac{dy}{dx} = (4 - y)(5 - y).$

Graph $\dfrac{dy}{dx}$ as a function of y, and use your graph to discuss the stability of the equilibria.

4 Suppose that $\dfrac{dy}{dx} = y(y - 1)(y - 2).$

Graph $\dfrac{dy}{dx}$ as a function of y, and use your graph to discuss the stability of the equilibria.