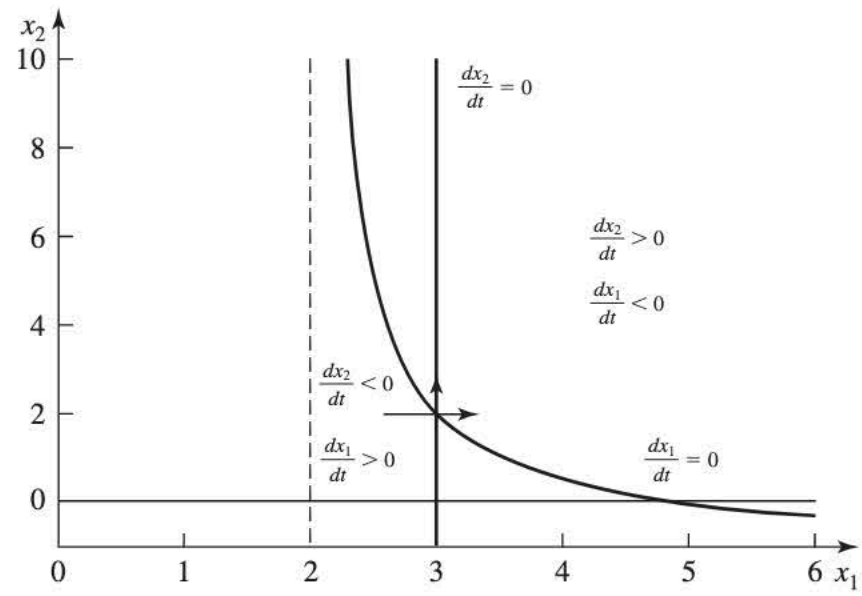

Figure 11.42 The zero isoclines in the x_1–x_2 plane.

the vertical arrow in the x_2-direction. Since the vertical arrow is on the zero isocline of x_2, the sign of dx_2/dt does not change as we cross the equilibrium in the x_2-direction. Therefore, $a_{22} = 0$. The signs of a_{12} and a_{21} follow from observing that if we cross the zero isocline of x_1 in the x_2-direction (the vertical arrow), then dx_1/dt changes from positive to negative, making $a_{12} < 0$. If we cross the zero isocline of x_2 in the direction of x_1 (the horizontal arrow), we see that dx_2/dt changes from negative to positive, making $a_{21} > 0$.

To determine the stability of $\hat{\mathbf{x}}$, we look at the trace and the determinant. Since the trace is negative and the determinant is positive, we conclude that the equilibrium is locally stable. ■

This simple graphical approach does not always give us the signs of the real parts of the eigenvalues, as illustrated in the following example: Suppose that we arrive at the Jacobi matrix in which the signs of the entries are

$$\begin{bmatrix} + & - \\ - & - \end{bmatrix}$$

The trace may now be positive or negative. Therefore, we cannot conclude anything about the eigenvalues. In this case, we would have to compute the eigenvalues or the trace and the determinant explicitly and cannot rely on the signs alone.

Section 11.3 Problems

■ 11.3.1

In Problems 1–6, the point (0, 0) is always an equilibrium. Use the analytical approach to investigate its stability.

1. $\frac{dx_1}{dt} = x_1 - 2x_2 + x_1x_2$
$\frac{dx_2}{dt} = -x_1 + x_2$

2. $\frac{dx_1}{dt} = -x_1 - x_2 + x_1^2$
$\frac{dx_2}{dt} = x_2 - x_1^2$

3. $\frac{dx_1}{dt} = x_1 + x_1^2 - 2x_1x_2 + x_2$
$\frac{dx_2}{dt} = x_1$

4. $\frac{dx_1}{dt} = 3x_1x_2 - x_1 + x_2$
$\frac{dx_2}{dt} = x_2^2 - x_1$

5. $\frac{dx_1}{dt} = x_1e^{-x_2}$
$\frac{dx_2}{dt} = 2x_2e^{x_1}$

6. $\frac{dx_1}{dt} = -2\sin x_1$
$\frac{dx_2}{dt} = -x_2e^{x_1}$

In Problems 7–12, find all equilibria of each system of differential equations and use the analytical approach to determine the stability of each equilibrium.

7. $\frac{dx_1}{dt} = -x_1 + 2x_1(1 - x_1)$
$\frac{dx_2}{dt} = -x_2 + 5x_2(1 - x_1 - x_2)$

8. $\frac{dx_1}{dt} = -x_1 + 3x_1(1 - x_1 - x_2)$
$\frac{dx_2}{dt} = -x_2 + 5x_2(1 - x_1 - x_2)$

9. $\frac{dx_1}{dt} = 4x_1(1 - x_1) - 2x_1x_2$
$\frac{dx_2}{dt} = x_2(2 - x_2) - x_2$

10. $\frac{dx_1}{dt} = 2x_1(5 - x_1 - x_2)$
$\frac{dx_2}{dt} = 3x_2(7 - 3x_1 - x_2)$

11. $$\frac{dx_1}{dt} = x_1 - x_2$$
$$\frac{dx_2}{dt} = x_1x_2 - x_2$$

12. $$\frac{dx_1}{dt} = x_1x_2 - x_2$$
$$\frac{dx_2}{dt} = x_1 + x_2$$

13. For which value of a has

$$\frac{dx_1}{dt} = x_2(x_1 + a)$$
$$\frac{dx_2}{dt} = x_2^2 + x_2 - x_1$$

a unique equilibrium? Characterize its stability.

14. Assume that $a > 0$. Find all point equilibria of

$$\frac{dx_1}{dt} = 1 - ax_1x_2$$
$$\frac{dx_2}{dt} = ax_1x_2 - x_2$$

and characterize their stability.

■ 11.3.2

15. Assume that

$$\frac{dx_1}{dt} = x_1(10 - 2x_1 - x_2)$$
$$\frac{dx_2}{dt} = x_2(10 - x_1 - 2x_2)$$

(a) Graph the zero isoclines.

(b) Show that $(\frac{10}{3}, \frac{10}{3})$ is an equilibrium, and use the analytical approach to determine its stability.

16. Assume that

$$\frac{dx_1}{dt} = x_1(10 - x_1 - 2x_2)$$
$$\frac{dx_2}{dt} = x_2(10 - 2x_1 - x_2)$$

(a) Graph the zero isoclines.

(b) Show that $(\frac{10}{3}, \frac{10}{3})$ is an equilibrium, and use the analytical approach to determine its stability.

In Problems 17–22, use the graphical approach for 2×2 systems to find the sign structure of the Jacobi matrix at the indicated equilibrium. If possible, determine the stability of the equilibrium. Assume that the system of differential equations is given by

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

Furthermore, assume that x_1 and x_2 are both nonnegative. In each problem, the zero isoclines are drawn and the equilibrium we want to investigate is indicated by a dot. Assume that both x_1 and x_2 increase close to the origin and that f_1 and f_2 change sign when crossing their zero isoclines.

17. See Figure 11.43.

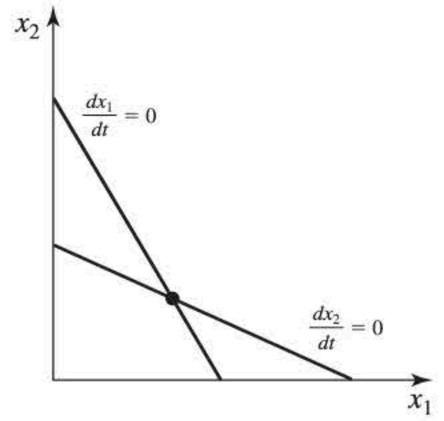

Figure 11.43

18. See Figure 11.44.

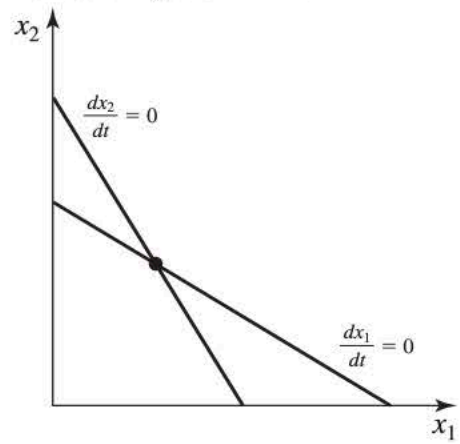

Figure 11.44

19. See Figure 11.45.

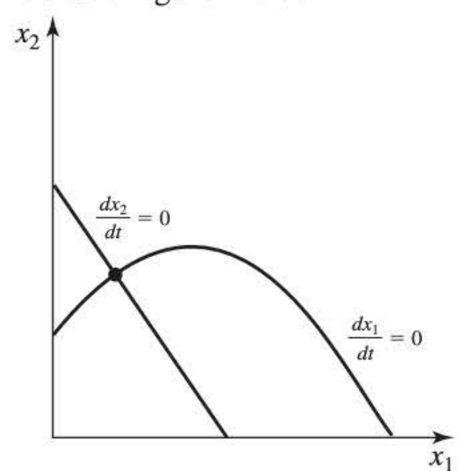

Figure 11.45

20. See Figure 11.46.

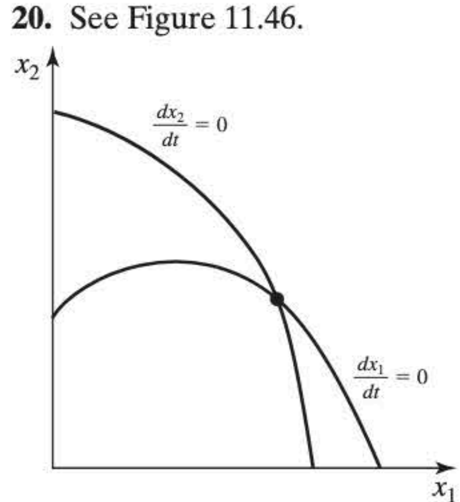

Figure 11.46

21. See Figure 11.47.

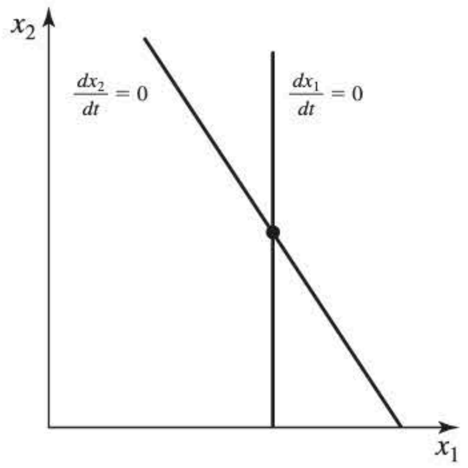

Figure 11.47

22. See Figure 11.48.

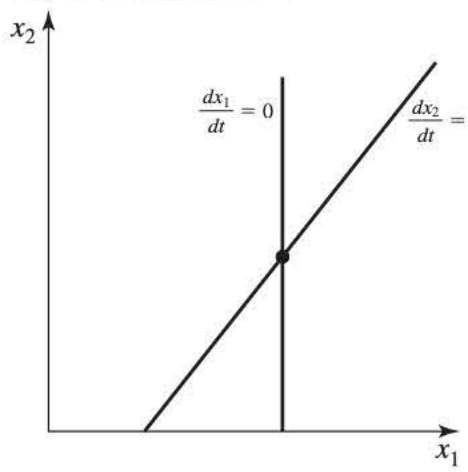

Figure 11.48

23. Let

$$\frac{dx_1}{dt} = x_1(2 - x_1) - x_1x_2$$
$$\frac{dx_2}{dt} = x_1x_2 - x_2$$

(a) Graph the zero isoclines.

(b) Show that (1, 1) is an equilibrium. Use the graphical approach to determine its stability.

24. Let

$$\frac{dx_1}{dt} = x_1(2 - x_1^2) - x_1x_2$$
$$\frac{dx_2}{dt} = x_1x_2 - x_2$$

(a) Graph the zero isoclines.

(b) Show that (1, 1) is an equilibrium. Use the graphical approach to determine its stability.

11.4 Nonlinear Systems: Applications

11.4.1 The Lotka–Volterra Model of Interspecific Competition

Imagine two species of plants growing together in the same plot. They both use similar resources: light, water, and nutrients. The use of these resources by one individual reduces their availability to other individuals. We call this type of interaction between individuals **competition**. **Intraspecific competition** occurs between individuals of the same species, **interspecific competition** between individuals of different species. Competition may result in reduced fecundity or reduced survivorship (or both). The effects of competition are often more pronounced when the number of competitors is higher.

In this subsection, we will discuss the Lotka–Volterra model of interspecific competition, which incorporates density-dependent effects of competition in the manner described previously. The model is an extension of the logistic equation to the case of two species. To describe it, we denote the population size of species 1 at time t by $N_1(t)$ and that of species 2 at time t by $N_2(t)$. Each species grows according to the logistic equation when the other species is absent. We denote their respective carrying capacities by K_1 and K_2, and their respective intrinsic rates of growth by r_1 and r_2. We assume that K_1, K_2, r_1, and r_2 are positive. In addition, the two species may have inhibitory effects on each other. We measure the effect of species 1 on species 2 by the **competition coefficient** α_{21}; the effect of species 2 on species 1 is measured by the competition coefficient α_{12}. The Lotka–Volterra model of interspecific competition is then given by the following system of differential