MA 138
Calculus II with Life Science Applications
FIRST MIDTERM

Spring 2024
02/06/2024

Name: Answer Key

Sect. #: ______________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■ [b] [c] [d] [e]

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. ■ [b] [c] [d] [e]

2. [a] ■ [c] [d] [e]

3. ■ [b] [c] [d] [e]

4. [a] [b] [c] ■ [e]

5. [a] ■ [c] [d] [e]

6. [a] [b] ■ [d] [e]

7. [a] [b] [c] [d] ■

8. [a] [b] ■ [d] [e]

9. [a] [b] [c] [d] ■

10. [a] [b] [c] ■ [e]

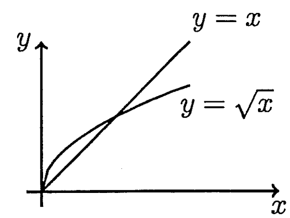
GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		50 pts
11.		10 pts
12.		10 pts
13.		10 pts
14.		10 pts
15.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Lecturer	Time/Location
001-004	Alberto Corso	MWF 10:00 am - 10:50 am, CB 212
Section #	**Recitation Instructor**	**Time/Location**
001	Ian Robinson	TR 09:00 am - 09:50 am, CB 307
002	Ian Robinson	TR 10:00 am - 10:50 am, CB 307

1. The area of the region enclosed by the curves $y = \sqrt{x}$ and $y = x$ is described by the following integral:



Possibilities:

(a) $\int_0^1 [\sqrt{x} - x]\, dx$

(b) $\int_0^1 [x\sqrt{x}]\, dx$

(c) $\int_0^1 [x - \sqrt{x}]\, dx$

(d) $\int_0^1 \sqrt{x}\, dx$

(e) None of the above

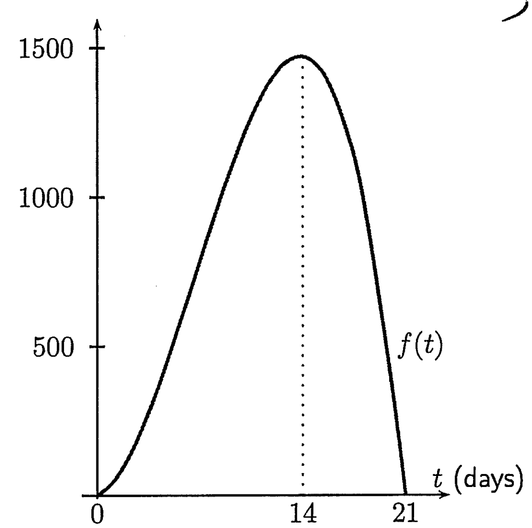
notice that the intersection points are

$$\begin{cases} y = \sqrt{x} \\ y = x \end{cases} \Longleftrightarrow \sqrt{x} = x$$

or $x = x^2 \Longleftrightarrow x(x-1) = 0$

$x = 0, 1$

2. Measles is a highly contagious infection of the respiratory system and is caused by a virus. The virus enters through the respiratory tract and replicates there before spreading into the bloodstream and then the skin. In a person with no immunity to measles the characteristic rash usually appears about two weeks after infection, at which point the virus reaches a peak density in the blood. The virus level then decreases fairly rapidly over the next few days as a result of the immune response.
This progression, or pathogenesis, of the disease is approximated in the pathogenesis curve $f(t)$ graphed on the side. This curve has been modeled by the polynomial



$$\begin{aligned} f(t) &= -t(t-21)(t+1) \\ &= -t^3 + 20t^2 + 21t, \end{aligned}$$

where t is measured in days and $f(t)$ is measured in the number of infected cell per mL of blood plasma. Over the course of the 21-day infection, what is the **average** level of infection? In other words, what is the **average** of the function $f(t)$ over the interval $[0, 21]$?

(a) ≈ 625.45 infected cell per mL of blood plasma

(b) ≈ 845.25 infected cell per mL of blood plasma

(c) $\approx 1{,}024.33$ infected cell per mL of blood plasma

(d) $\approx 1{,}200.75$ infected cell per mL of blood plasma

(e) $\approx 1{,}500$ infected cell per mL of blood plasma

We are asked in #2 to compute the average of the function

$$f(t) = -t(t-21)(t+1)$$

over the interval $[0, 21]$. Thus

$$f_{ave} = \frac{1}{21-0}\int_0^{21} -t(t-21)(t+1)\,dt$$

$$= \frac{1}{21}\int_0^{21}(-t^3+20t^2+21t)\,dt$$

$$= \frac{1}{21}\left[-\frac{1}{4}t^4 + \frac{20}{3}t^3 + \frac{21}{2}t^2\Big|_0^{21}\right]$$

$$= \frac{1}{21}\left[-\frac{21^4}{4} + \frac{20}{3}(21)^3 + \frac{21}{2}(21)^2 - 0\right]$$

$$= \frac{-21^3}{4} + 20\cdot 21\cdot 7 + \frac{21^2}{2}$$

$$= \frac{-21^3 + 20\cdot 21\cdot 28 + 2\cdot 21^2}{4} = \frac{3381}{4}$$

3. Find $\int \sin(x)\cos^2(x)\,dx$ using the substitution $u = \cos(x)$.

$u = \cos(x)$ $\quad \frac{du}{dx} = -\sin x$ $\quad$ so $-du = \sin(x)dx$

Thus $\int \sin(x)\cos^2(x)\,dx = \int -u^2\,du = -\frac{1}{3}u^3 + C$

$= \boxed{-\frac{1}{3}\cos^3(x) + C}$

Possibilities:

(a) $\boxed{-\frac{\cos^3(x)}{3} + C}$

(b) $-\frac{\cos^6(x)}{6} + C$

(c) $-\frac{\sin^3(x)}{3} + C$

(d) $\frac{\cos^2(x)}{2} + C$

(e) $-\cos(x)\sin^2(x) + C$

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4. Evaluate $\int \frac{x}{e^{x^2}}\,dx$ using the substitution $u = x^2$

$\frac{du}{dx} = 2x$ $\quad$ So $\quad \frac{1}{2}du = x\,dx$ $\quad$ and then integral becomes $\int \frac{\frac{1}{2}du}{e^u}$

OR $= +\frac{1}{2}\int e^{-u}\,du$

$= -\frac{1}{2}e^{-u} + C$

$= -\frac{1}{2}e^{-x^2} + C = \boxed{-\frac{1}{2e^{x^2}} + C}$

Possibilities:

(a) $\frac{1}{2e^{x^2}} + C$

(b) $\frac{1}{e^{x^2}} + C$

(c) $-\frac{1}{e^{x^2}} + C$

(d) $\boxed{-\frac{1}{2e^{x^2}} + C}$

(e) $\frac{x^2}{2e^{x^3}} + C$

5. Let f be a function defined on the interval $[2,7]$ with continuous derivatives. Suppose in addition that we are given the following information about f and f'

$$f(2)=5 \qquad f(7)=10 \qquad f'(7)=3.$$

Find the value of

$$\int_2^7 (2x-4)f''(x)\,dx$$

$$=(2x-4)f'(x)\Big|_2^7 - \int_2^7 2\cdot f'(x)\,dx =$$

$$=(2\cdot 7-4)f'(7) - (2\cdot 2-4)f'(2) - 2\int_2^7 f'(x)\,dx$$

$$=10\cdot 3 - \underset{0}{\cancel{0\cdot f'(2)}} - 2\left[f(x)\Big|_2^7\right] =$$

$$=30-2(10-5)$$

$$=20$$

Possibilities:

(a) 25

$\boxed{\text{(b) } 20}$

(c) 10

(d) 0

(e) The value can't be determined without knowing $f'(2)$

6. Which of the following is equal to the integral

$$\int x\cos(2x)\,dx?$$

$$=x\cdot\left[\frac{1}{2}\sin(2x)\right] - \int 1\cdot\frac{1}{2}\sin(2x)\,dx$$

$$=\frac{x}{2}\sin(2x) + \frac{1}{2}\int -\sin(2x)\,dx$$

$$=\frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C$$

Possibilities:

(a) $-\frac{x}{2}\sin(2x)+\frac{1}{4}\cos(2x)+C$

(b) $\frac{x}{2}\sin(2x)-\frac{1}{4}\cos(2x)+C$

$\boxed{\text{(c) } \frac{x}{2}\sin(2x)+\frac{1}{4}\cos(2x)+C}$

(d) $-2x\sin(2x)+\cos(2x)+C$

(e) $-2x\sin(2x)-4\cos(2x)+C$

7. Which of the following is the correct form of the partial fraction expansion of

$$\frac{841}{(x-4)^2(x^2+x+9)}?$$

[**Hint:** Observe that x^2+x+9 is an irreducible quadratic polynomial.]

(a) $\frac{Ax+B}{(x-4)^2}+\frac{Cx+D}{x^2+x+9}$

(b) $\frac{A}{(x-4)^2}+\frac{Bx+C}{x^2+x+9}$

(c) $\frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{C}{x^2+x+9}$

(d) $\frac{A}{(x-4)^2}+\frac{B}{x^2+x+9}$

$\boxed{\text{(e) } \frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{Cx+D}{x^2+x+9}}$

You can verify that

$$= \frac{-9}{x-4}+\frac{29}{(x-4)^2}+\frac{9x+16}{x^2+x+9}$$

8. Which of the following represents

$$\int \frac{3x+4}{x^2+3x+2}\,dx?$$

$$\frac{3x+4}{x^2+3x+2} = \frac{3x+4}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

need: $3x+4 = A(x+2)+B(x+1)$

set $\boxed{x=-1} \Longrightarrow 3(-1)+4 = A(1)+B\cdot\cancel{0}$ $\therefore A=1$

set $\boxed{x=-2} \Longrightarrow 3(-2)+4 = A\cdot\cancel{(0)}+B(-1)$ or $-2=-B$ $\therefore \boxed{B=2}$

Possibilities:

(a) $(3/2x^2+4x)\ln|x^2+3x+2|+C$

(b) $3/2\ln|x^2+3x+2|+C$

$\boxed{\text{(c) } \ln|x+1|+2\ln|x+2|+C}$

(d) $2\ln|x+1|+\ln|x+2|+C$

(e) $\ln|x+1|+\ln|x+2|+C$

Thus

$$\int\left(\frac{1}{x+1}+\frac{2}{x+2}\right)dx = \boxed{\ln|x+1|+2\ln|x+2|+C}$$

9. Evaluate the following integral

$$\int_{-1}^{6} \frac{1}{x-1}\,dx$$

set $u = x-1$ so that $du = dx$. Hence the integral becomes $\int_{-2}^{5} \frac{1}{u}\,du =$

$$= \underbrace{\int_{-2}^{0} \frac{1}{u}\,du}_{} + \underbrace{\int_{0}^{5} \frac{1}{u}\,du}_{}$$

both integrals diverge

Possibilities:

(a) $\ln(5) - \ln(-2)$

(b) $\ln(5) - \ln(2)$

(c) $-\ln(5) + \ln(2)$

(d) $\ln(5) + \ln(2)$

(e) The integral diverges

10. Complete correctly the statement below

"The integral $\int_{1}^{\infty} \frac{1}{x^p} dx$ ..."

Possibilities:

(a) "...diverges, because it is improper"

(b) "...diverges if $p > 1$ and converges if $0 < p \leq 1$"

(c) "...converges if $p \geq 1$ and diverges if $0 < p < 1$"

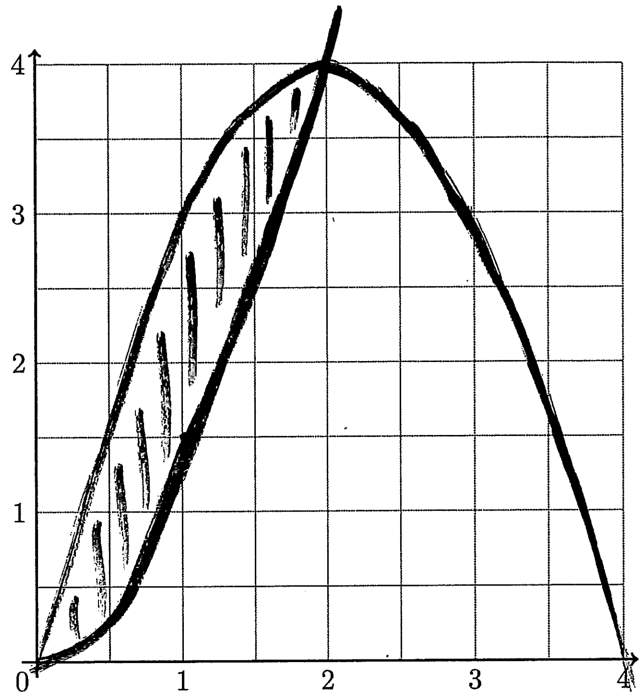
(d) "...converges if $p > 1$ and diverges if $0 < p \leq 1$

(e) "...diverges if p is negative and converges if p is positive"

11. Sketch the region enclosed by the parabolas

$$y = x^2 \qquad \text{and} \qquad y = 4x - x^2.$$

Then find the area of this region.



We first need to find the intersection points of the two parabolas

$$\begin{cases} y = x^2 \\ y = 4x - x^2 \end{cases}$$

$$\rightsquigarrow \quad x^2 = y = 4x - x^2$$

OR $\quad x^2 + x^2 - 4x = 0$

$$2x^2 - 4x = 0 \qquad 2x(x-2) = 0 \Rightarrow \boxed{x = 0, 2}$$

Hence the area under consideration is

$$A = \int_0^2 \left((4x - x^2) - x^2\right) dx = \int_0^2 (4x - 2x^2)\, dx$$

$$= 2x^2 - \frac{2}{3}x^3 \Big|_0^2 = \left[2(2)^2 - \frac{2}{3}(2)^3\right] - 0$$

$$= 8 - \frac{16}{3} = \frac{24-16}{3} = \boxed{\frac{8}{3} \simeq 2.667}$$

pts: /10

12. Use substitution to evaluate the definite integral $\int x\sqrt{x-1}\,dx$.

Clearly indicate your answer and the steps used to arrive at that answer. An unsupported answer will receive no credit.

set $u = x-1$ so that $\frac{du}{dx} = 1$ OR $du = dx$. Finally $x = u+1$.

Thus

$$\int x\sqrt{x-1}\,dx = \int (u+1)\sqrt{u}\,du = \int \left(u\sqrt{u} + \sqrt{u}\right)du$$

$$= \int \left(u^{3/2} + u^{1/2}\right)du = \frac{1}{3/2+1}u^{3/2+1} + \frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1} + C$$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}$$

$$= \frac{2}{15}(x-1)^{3/2}\left(3(x-1)+5\right) + C$$

$$= \frac{2}{15}(x-1)^{3/2}(3x+2) + C$$

pts: /10

13. Use integration by parts to evaluate the integral $\int (2x+3)e^x\,dx$.

Clearly indicate your answer and the steps used to arrive at that answer. An unsupported answer will receive no credit.

$$\int (2x+3)e^x\,dx = (2x+3)e^x - \int 2\cdot e^x\,dx$$

$$= (2x+3)e^x - 2e^x + C$$

$$= \left[(2x+3) - 2\right]e^x + C$$

$$= \boxed{(2x+1)e^x + C}$$

pts: /10

14. Evaluate the integral $\int \frac{4x-2}{x^3-x}\,dx$ by following the steps below:

(*a*) Completely factor the denominator $x^3 - x$.

(*b*) Compute the partial fraction decomposition of $\frac{4x-2}{x^3-x}$.

(*c*) Evaluate the integral.

Note that $x^3 - x = x(x^2-1) = x(x-1)(x+1)$

Thus $$\frac{4x-2}{x^3-x} = \frac{4x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

Hence we need :

$$\boxed{4x-2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}$$

using the cover-up method :

$x=0 \implies -2 = A(-1) + \cancel{B}\,0 + \cancel{C}\,0$ $\boxed{A=2}$

$x=1 \implies 2 = \cancel{A}\,0 + B\cdot 2 + \cancel{C}\,0$ $\boxed{B=1}$

$x=-1 \implies -6 = \cancel{A}\,0 + \cancel{B}\,0 + C\cdot 2$ $\boxed{C=-3}$

$$\int \frac{4x-2}{x^3-x}\,dx = \int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1}\right)dx =$$

$$\boxed{= 2\ln|x| + \ln|x-1| - 3\ln|x+1| + C}$$

$$= \ln\left[\frac{x^2|x-1|}{|x+1|^3}\right] + C$$

pts: /10

15. Integrate the improper integral $\int_0^{\infty} xe^{-x^2}\,dx.$

Clearly indicate your answer and the steps used to arrive at that answer. An unsupported answer will receive no credit.

$$\int_0^{\infty} x e^{-x^2} dx = \lim_{b\to\infty} \int_0^b -\frac{1}{2}(-2x)\, e^{-x^2} dx$$

$$= \lim_{b\to\infty} -\frac{1}{2}\int_0^b (-2x) e^{-x^2} dx =$$

$$= -\frac{1}{2} \lim_{b\to\infty} \int_0^{-b^2} e^{u}\, du$$

$$= -\frac{1}{2}\left(\lim_{b\to\infty} e^{u}\Big|_0^{-b^2} \right) =$$

$$= -\frac{1}{2} \lim_{b\to\infty} \left(\underbrace{e^{-b^2}}_{\to 0} - e^{0} \right) =$$

$$= -\frac{1}{2}(-1) = \boxed{\frac{1}{2}}$$

pts:	/10

Bonus. Evaluate the integral: $\displaystyle\int_1^4 e^{\sqrt{x}}\,dx.$

[**Hint:** Use the substitution $u = \sqrt{x}$ and then use integration by parts.]

$\boxed{u = \sqrt{x}}$ $\quad \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$ so

$2\sqrt{x}\,du = dx \quad$ OR $\quad \boxed{2u\,du = dx}$

Thus $\displaystyle\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 2u e^u\,du$

Now use integration by parts

$= 2ue^u\Big|_1^2 - \displaystyle\int_1^2 2\cdot e^u\,du =$

$= (2\cdot 2\cdot e^2 - 2e) - \left(2e^u\Big|_1^2\right) =$

$= 4e^2 - \cancel{2e} - (2e^2 - \cancel{2e}) = \boxed{2e^2}$

≈ 14.7781

pts: /10